Introduction to Algorithms

6.046J/18.401J/SMA5503

Lecture 16

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Graphs (review)

Definition. A ***directed graph (digraph)*** $G = (V, E)$ is an ordered pair consisting of

- a set V of ***vertices*** (singular: ***vertex***),
- a set $E \subseteq V \times V$ of ***edges***.

In an ***undirected graph*** $G = (V, E)$, the edge set E consists of *unordered* pairs of vertices.

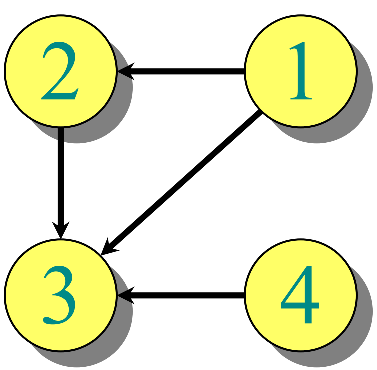
In either case, we have $|E| = O(V^2)$. Moreover, if G is connected, then $|E| \geq |V| - 1$, which implies that $\lg |E| = \Theta(\lg V)$.

(Review CLRS, Appendix B.)

Adjacency-matrix representation

The ***adjacency matrix*** of a graph $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$, is the matrix $A[1 \,.\,.\, n, 1 \,.\,.\, n]$ given by

$$A[i, j] = \begin{cases} 1 & \text{if } (i, j) \in E, \\ 0 & \text{if } (i, j) \notin E. \end{cases}$$

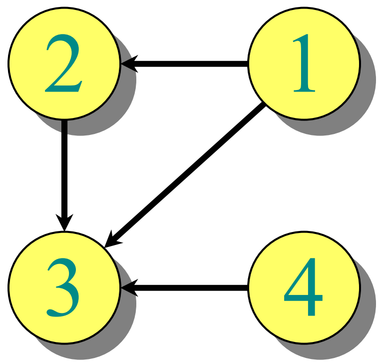


A	1	2	3	4
1	0	1	1	0
2	0	0	1	0
3	0	0	0	0
4	0	0	1	0

$\Theta(V^2)$ storage $\Rightarrow$ ***dense*** representation.

Adjacency-list representation

An ***adjacency list*** of a vertex $v \in V$ is the list $Adj[v]$ of vertices adjacent to v.



$Adj[1] = \{2, 3\}$
$Adj[2] = \{3\}$
$Adj[3] = \{\}$
$Adj[4] = \{3\}$

For undirected graphs, $|Adj[v]| = degree(v)$.
For digraphs, $|Adj[v]| = out\text{-}degree(v)$.

Handshaking Lemma: $\sum_{v \in V} = 2|E|$ for undirected graphs $\Rightarrow$ adjacency lists use $\Theta(V + E)$ storage — a ***sparse*** representation (for either type of graph).

Minimum spanning trees

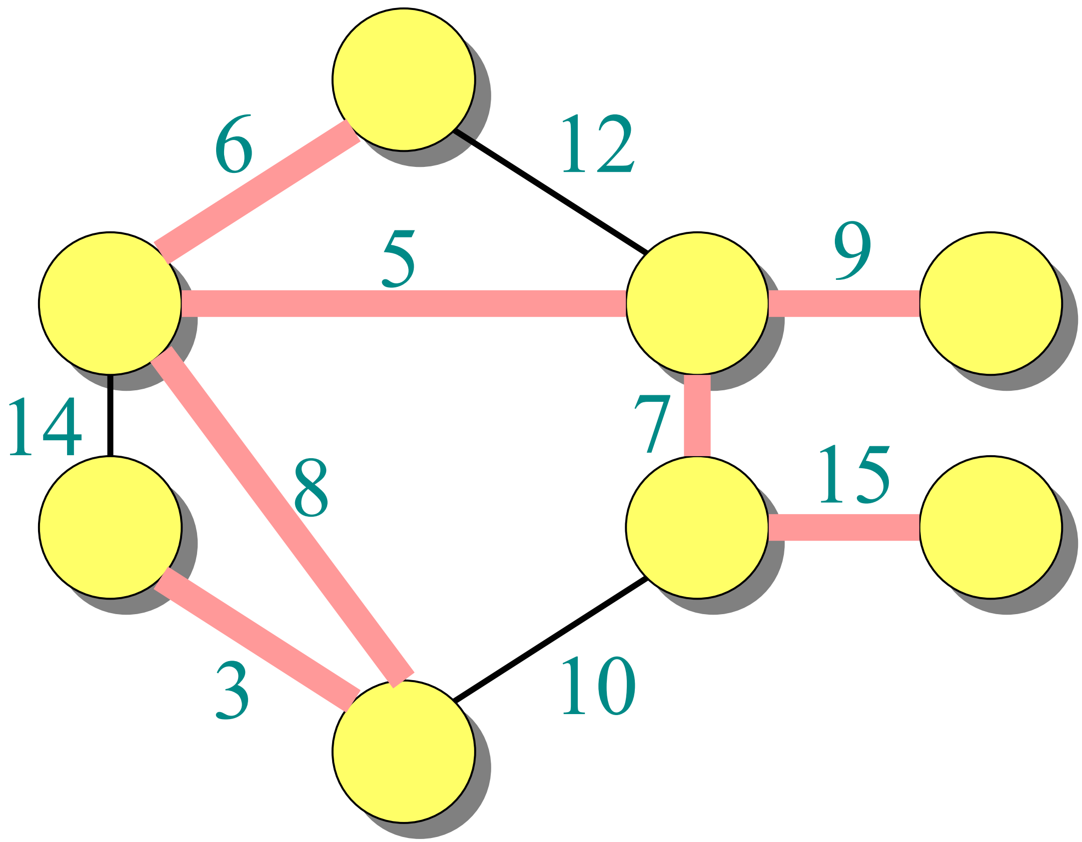
Input: A connected, undirected graph $G = (V, E)$ with weight function $w : E \to \mathbb{R}$.

- For simplicity, assume that all edge weights are distinct. (CLRS covers the general case.)

Output: A ***spanning tree*** T — a tree that connects all vertices — of minimum weight:

$$w(T) = \sum_{(u,v) \in T} w(u,v) .$$

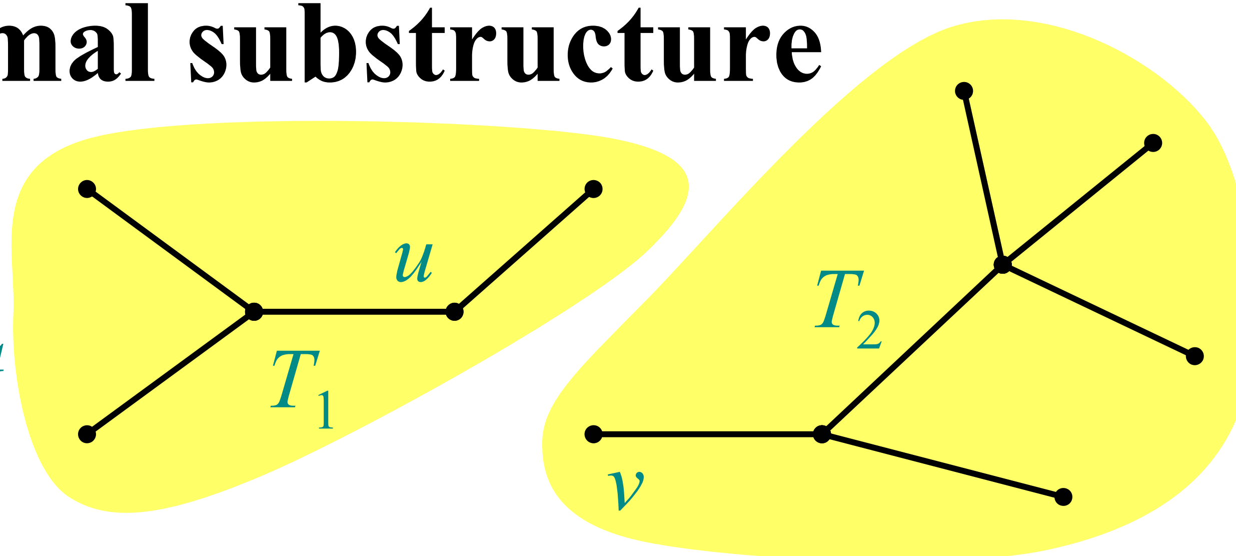
Example of MST



Optimal substructure

MST T:

(Other edges of G are not shown.)



Remove any edge $(u, v) \in T$. Then, T is partitioned into two subtrees T_1 and T_2.

Theorem. The subtree T_1 is an MST of $G_1 = (V_1, E_1)$, the subgraph of G ***induced*** by the vertices of T_1:

$$V_1 = \text{vertices of } T_1,$$
$$E_1 = \{ (x, y) \in E : x, y \in V_1 \}.$$

Similarly for T_2.

Proof of optimal substructure

Proof. Cut and paste:

$$w(T) = w(u, v) + w(T_1) + w(T_2).$$

If T_1' were a lower-weight spanning tree than T_1 for G_1, then $T' = \{(u, v)\} \cup T_1' \cup T_2$ would be a lower-weight spanning tree than T for G. $\square$

Do we also have overlapping subproblems?

- Yes.

Great, then dynamic programming may work!

- Yes, but MST exhibits another powerful property which leads to an even more efficient algorithm.

Hallmark for "greedy" algorithms

Greedy-choice property

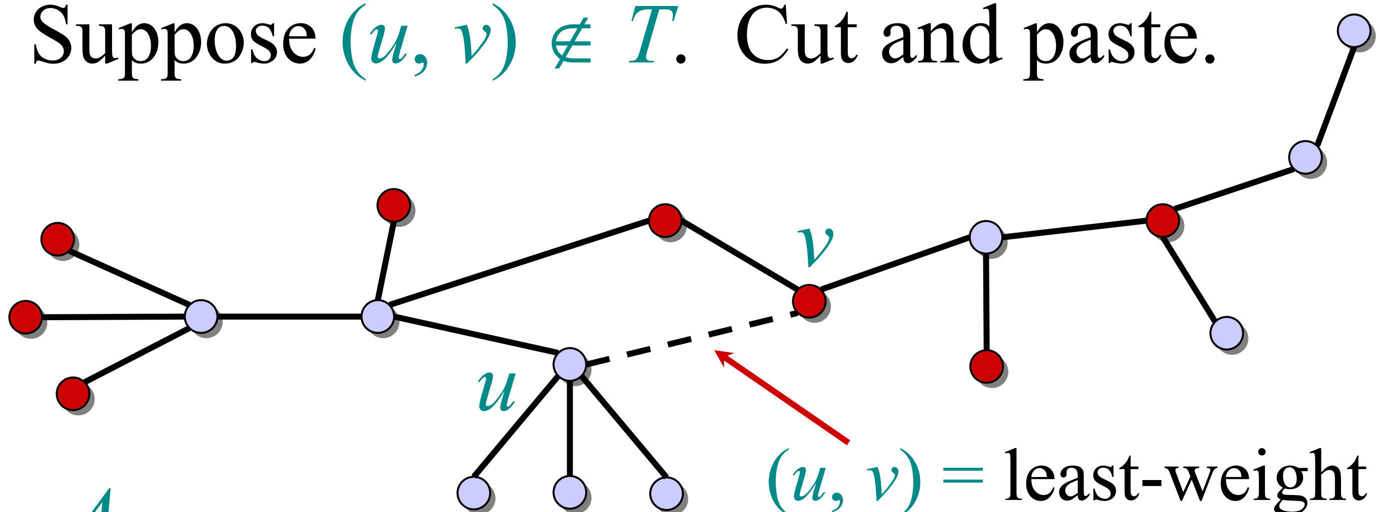
A locally optimal choice is globally optimal.

Theorem. Let T be the MST of $G = (V, E)$, and let $A \subseteq V$. Suppose that $(u, v) \in E$ is the least-weight edge connecting A to $V - A$. Then, $(u, v) \in T$.

Proof of theorem

Proof. Suppose $(u, v) \notin T$. Cut and paste.

T:



$\in A$

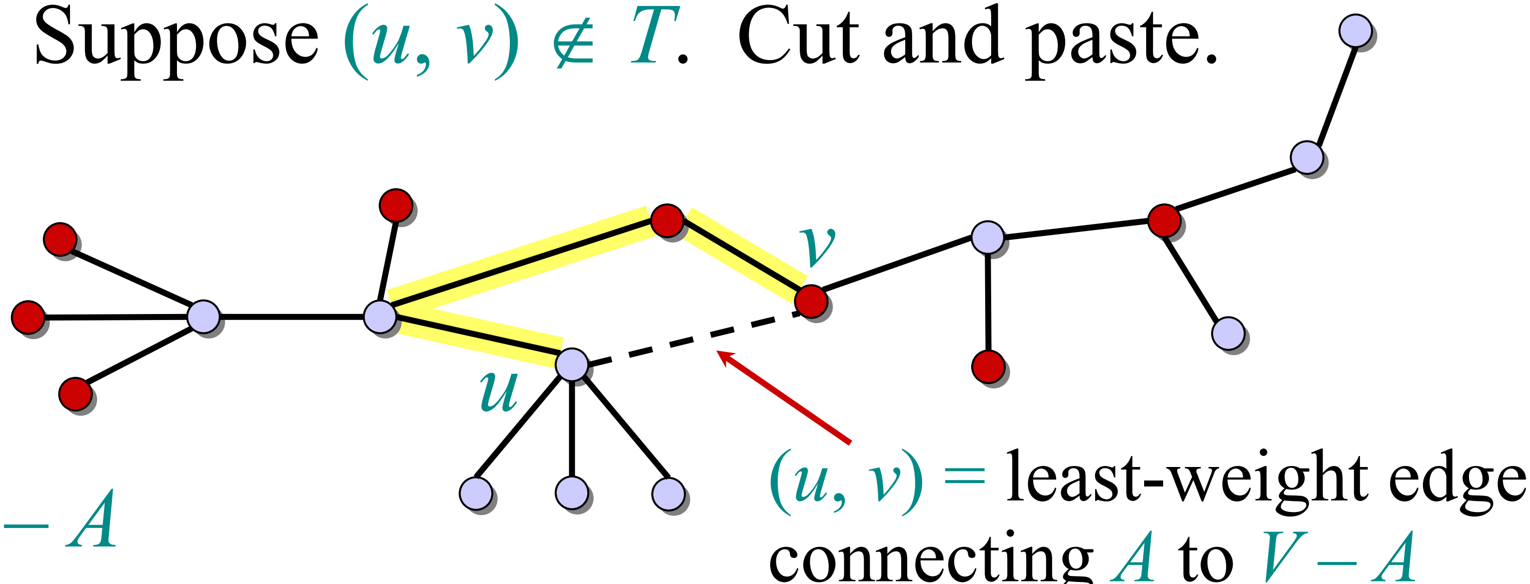
$\in V - A$

(u, v) = least-weight edge connecting A to $V - A$

Proof of theorem

Proof. Suppose $(u, v) \notin T$. Cut and paste.

T:



$\in A$

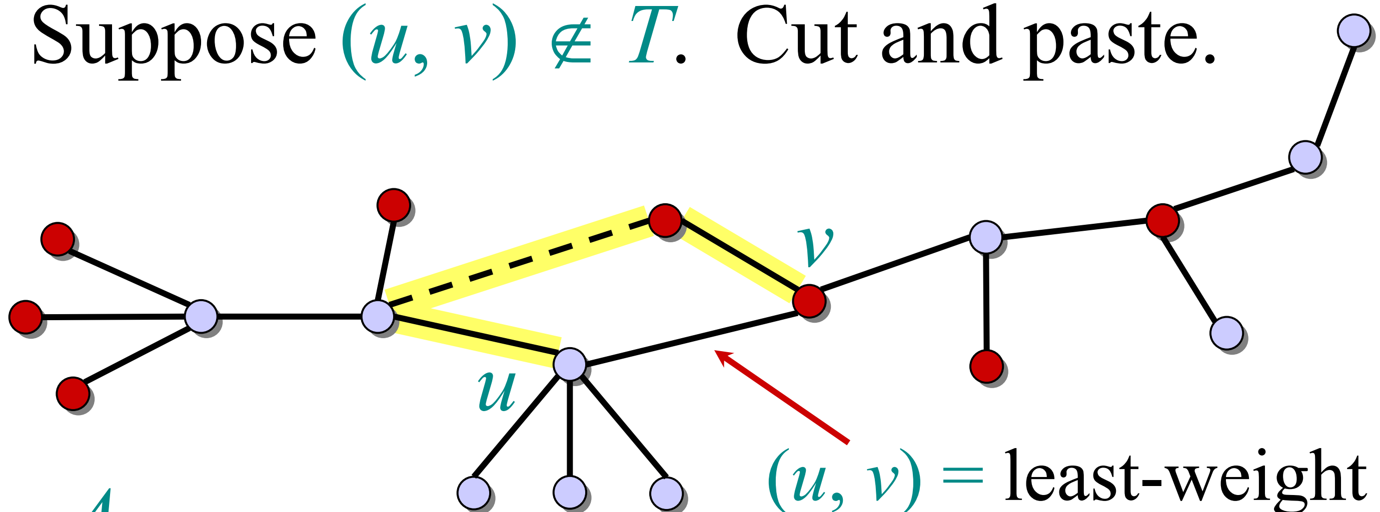
$\in V - A$

Consider the unique simple path from u to v in T.

Proof of theorem

Proof. Suppose $(u, v) \notin T$. Cut and paste.

T:



$\in A$

$\in V - A$

(u, v) = least-weight edge connecting A to $V - A$

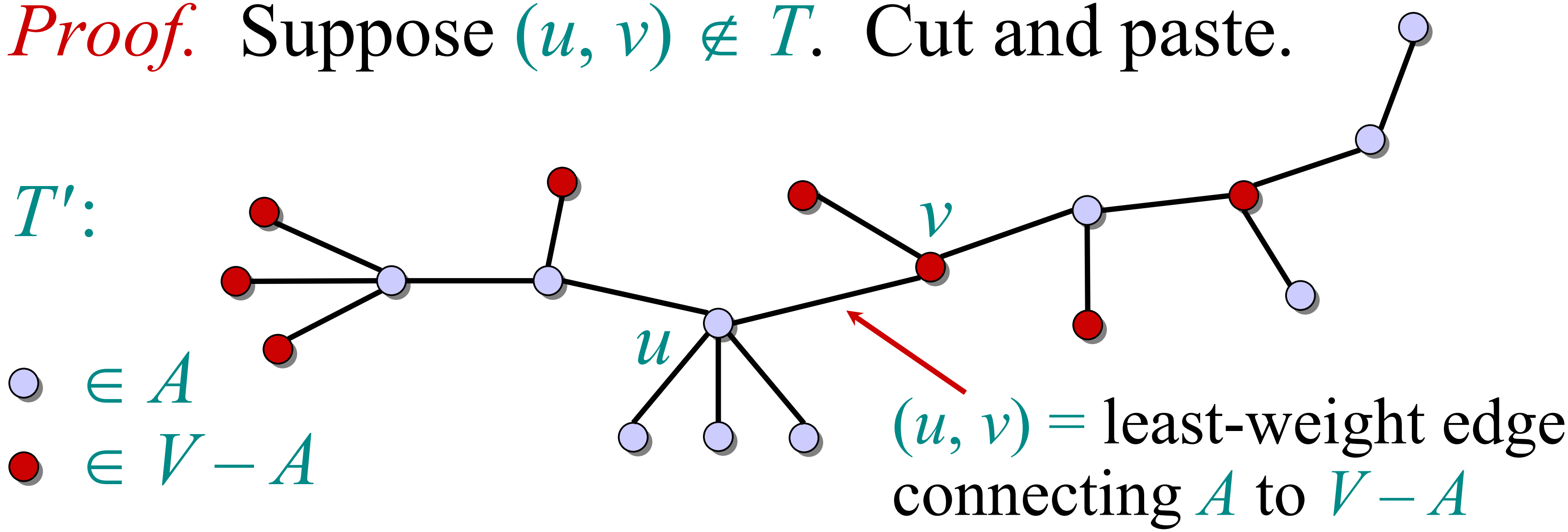
Consider the unique simple path from u to v in T.

Swap (u, v) with the first edge on this path that connects a vertex in A to a vertex in $V - A$.

Proof of theorem

Proof. Suppose $(u, v) \notin T$. Cut and paste.

T':



Consider the unique simple path from u to v in T.

Swap (u, v) with the first edge on this path that connects a vertex in A to a vertex in $V - A$.

A lighter-weight spanning tree than T results. $\square$

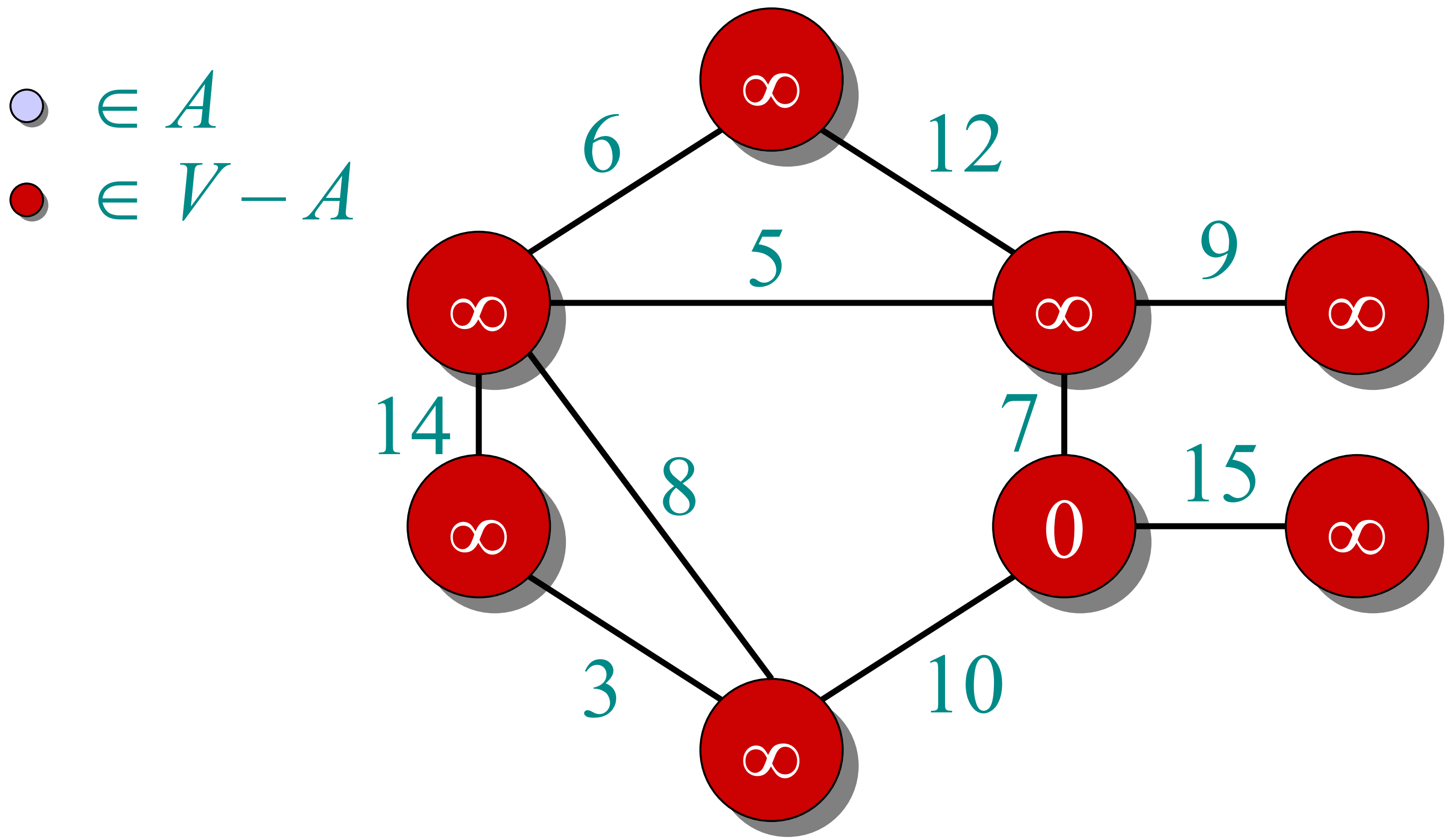
Prim's algorithm

IDEA: Maintain $V - A$ as a priority queue Q. Key each vertex in Q with the weight of the least-weight edge connecting it to a vertex in A.

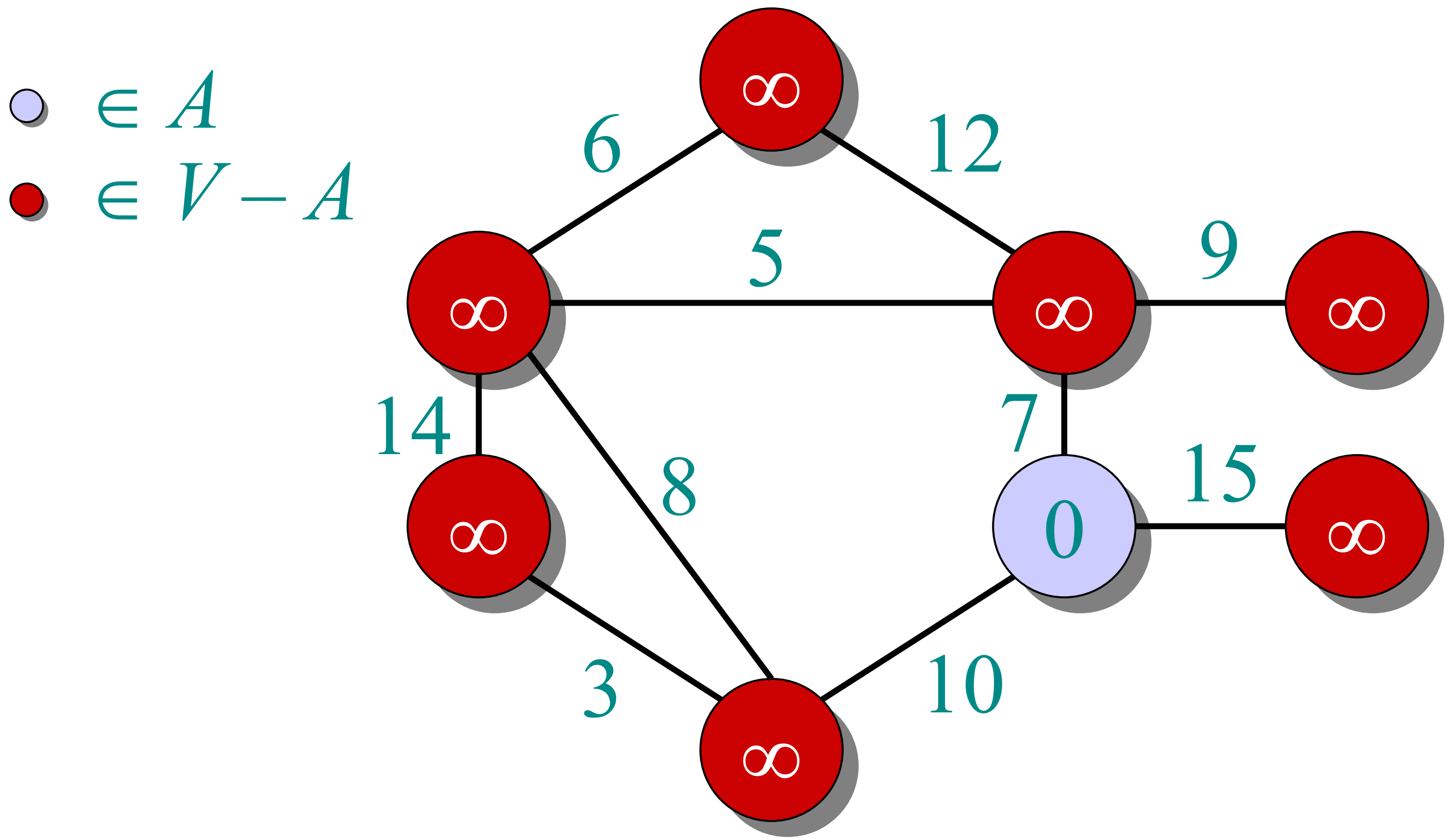
$Q \leftarrow V$
$key[v] \leftarrow \infty$ for all $v \in V$
$key[s] \leftarrow 0$ for some arbitrary $s \in V$
while $Q \neq \emptyset$
  do $u \leftarrow$ EXTRACT-MIN(Q)
    for each $v \in Adj[u]$
      do if $v \in Q$ and $w(u, v) < key[v]$
        then $key[v] \leftarrow w(u, v)$ ▷ DECREASE-KEY
          $\pi[v] \leftarrow u$

At the end, $\{(v, \pi[v])\}$ forms the MST.

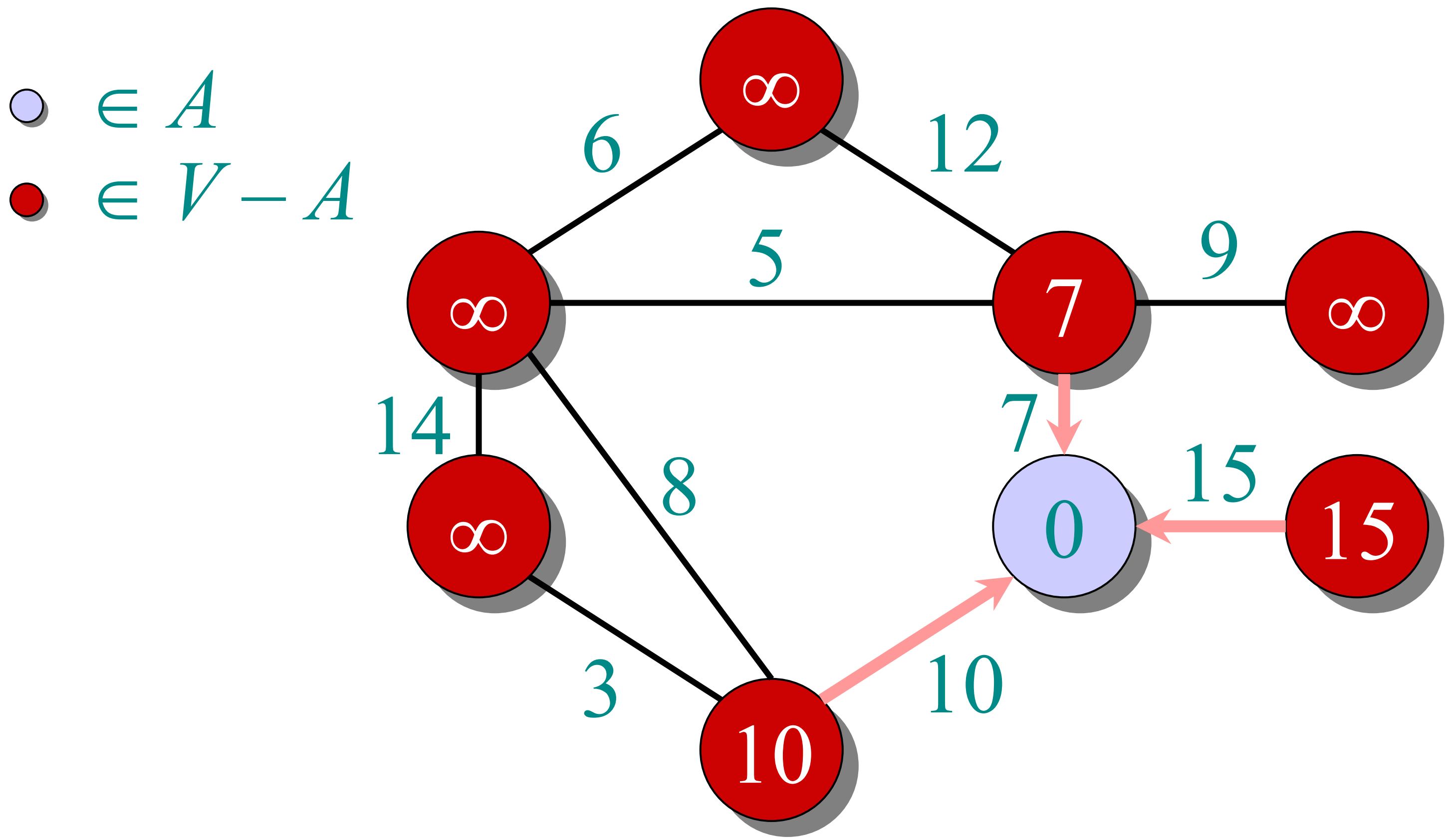
Example of Prim's algorithm



Example of Prim's algorithm



Example of Prim's algorithm



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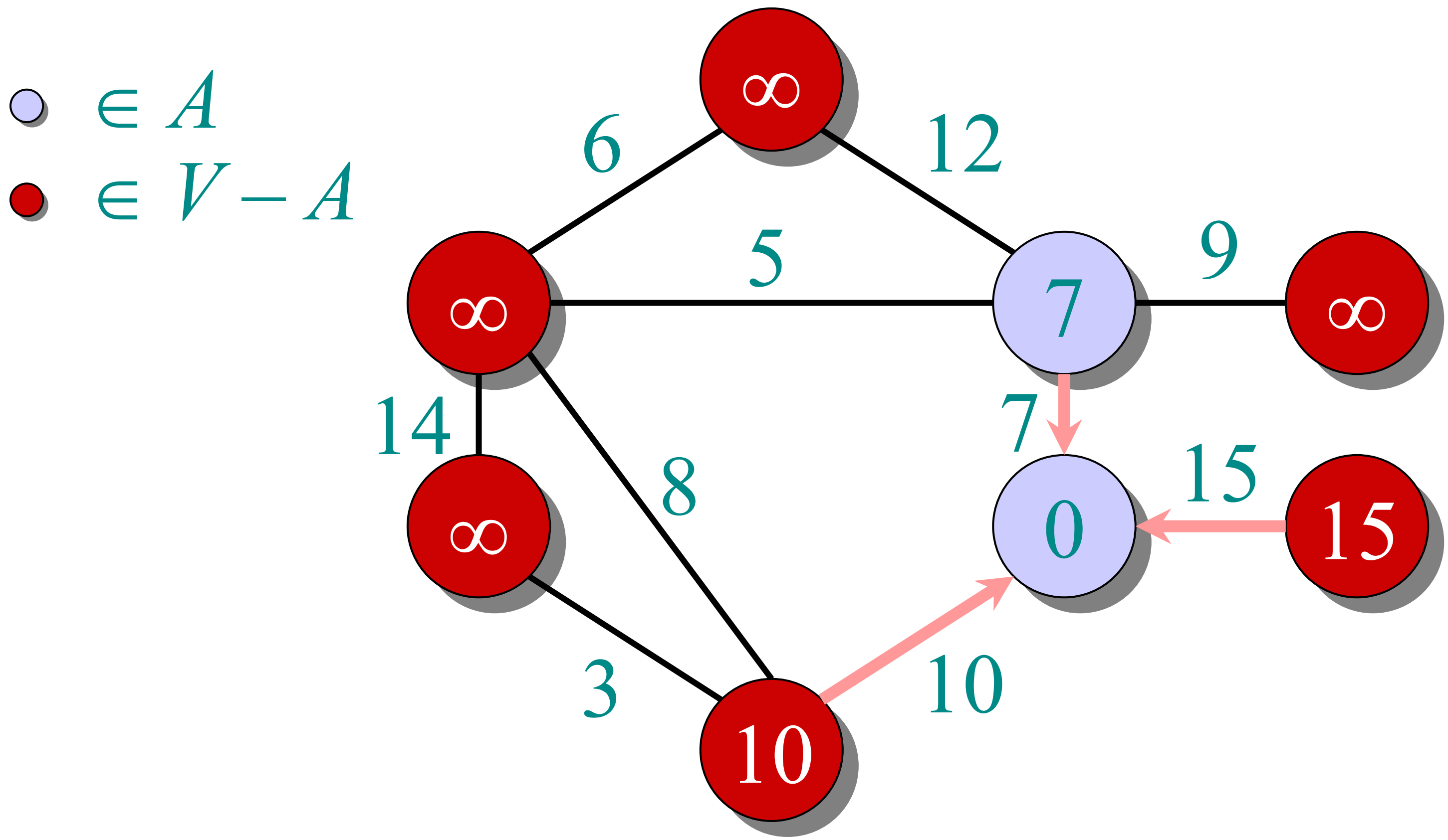

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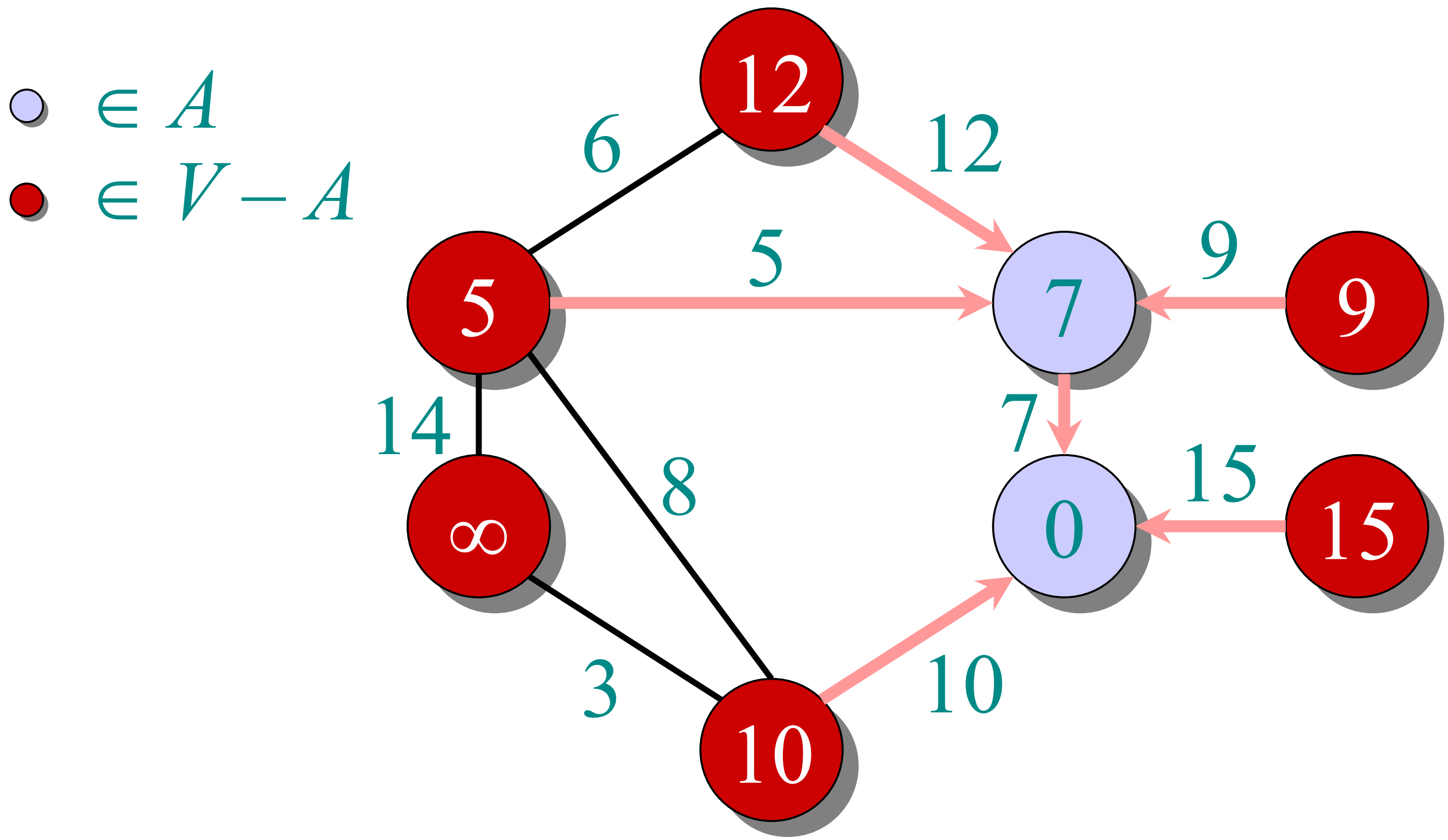

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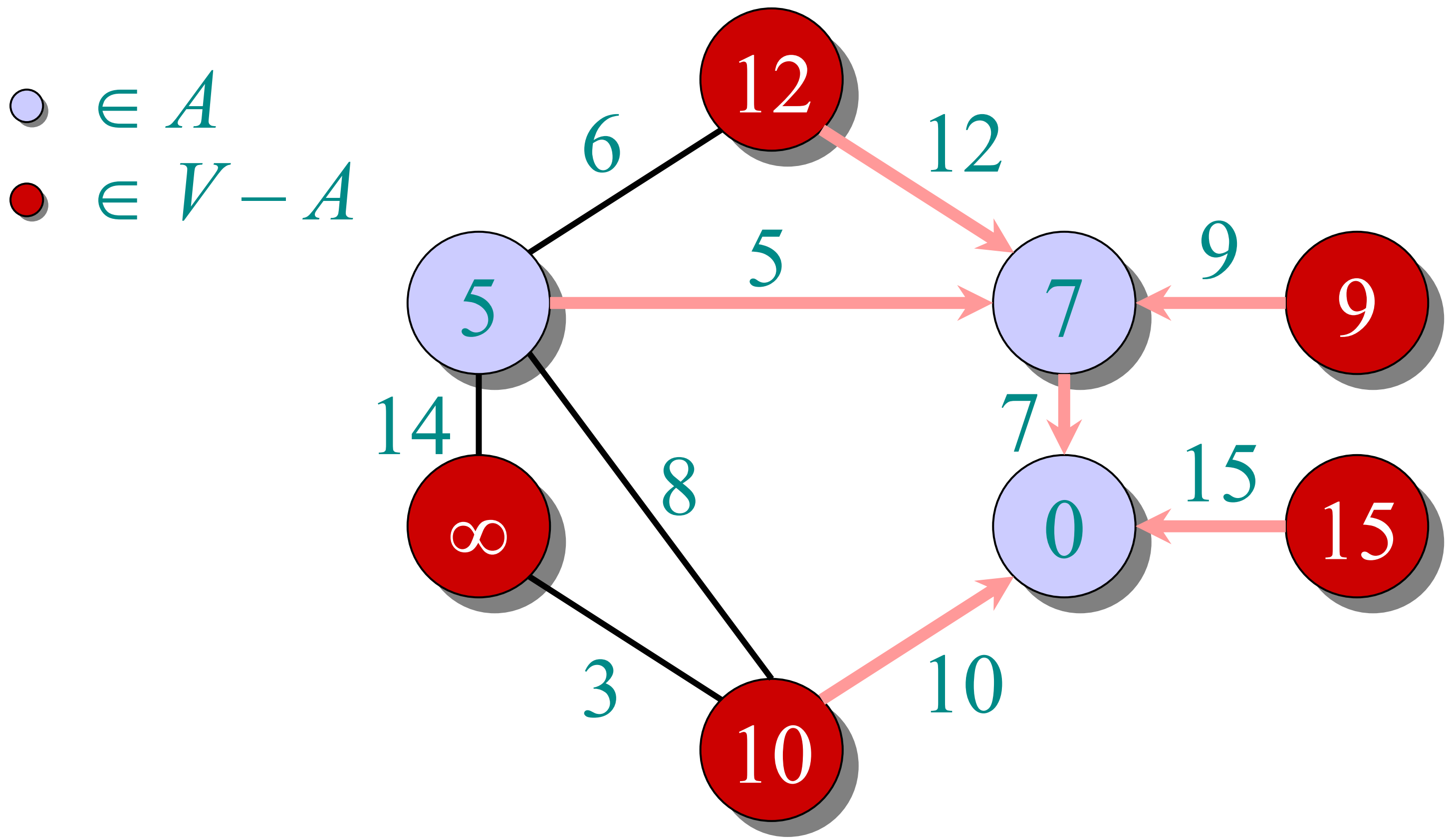

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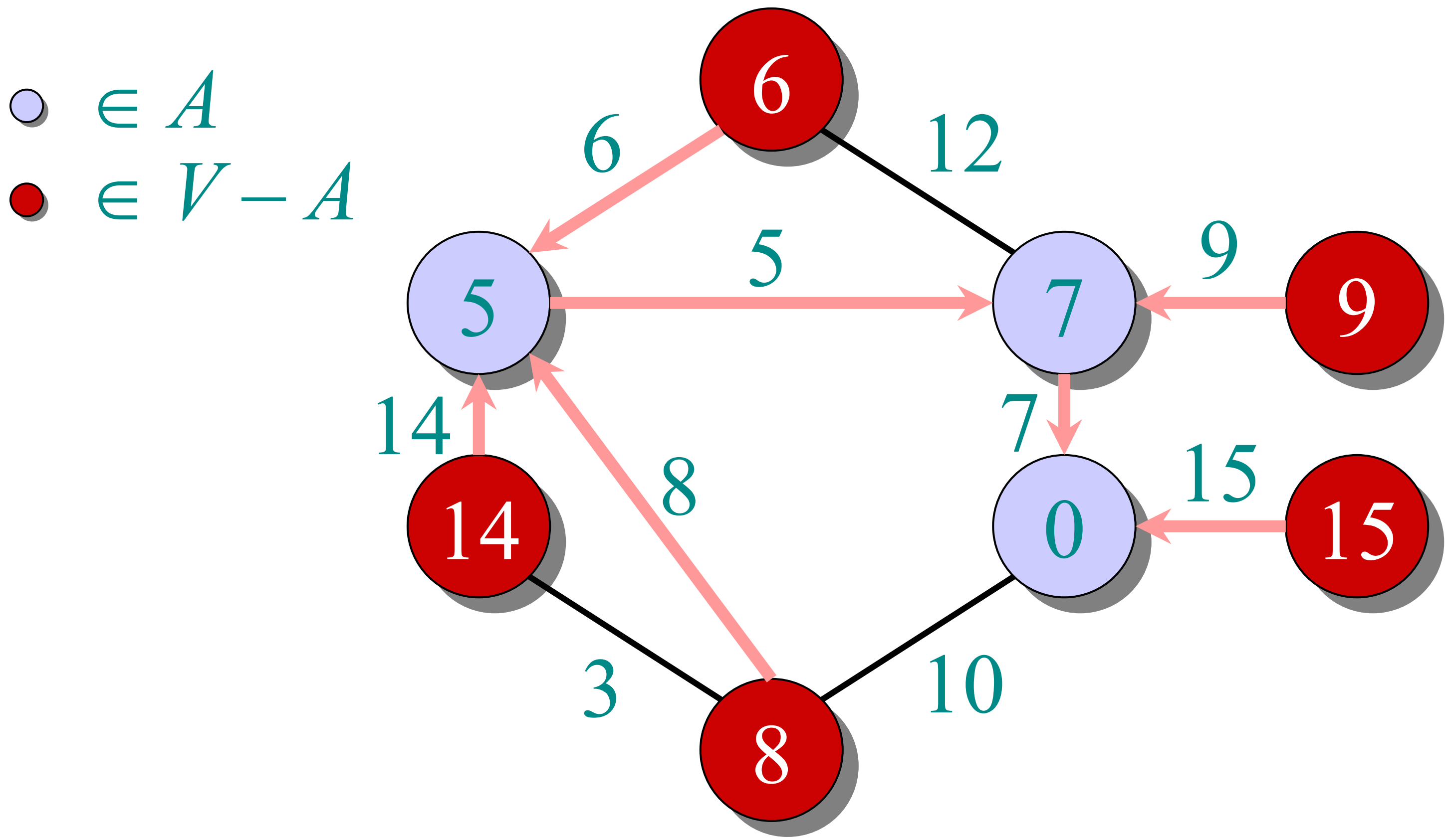
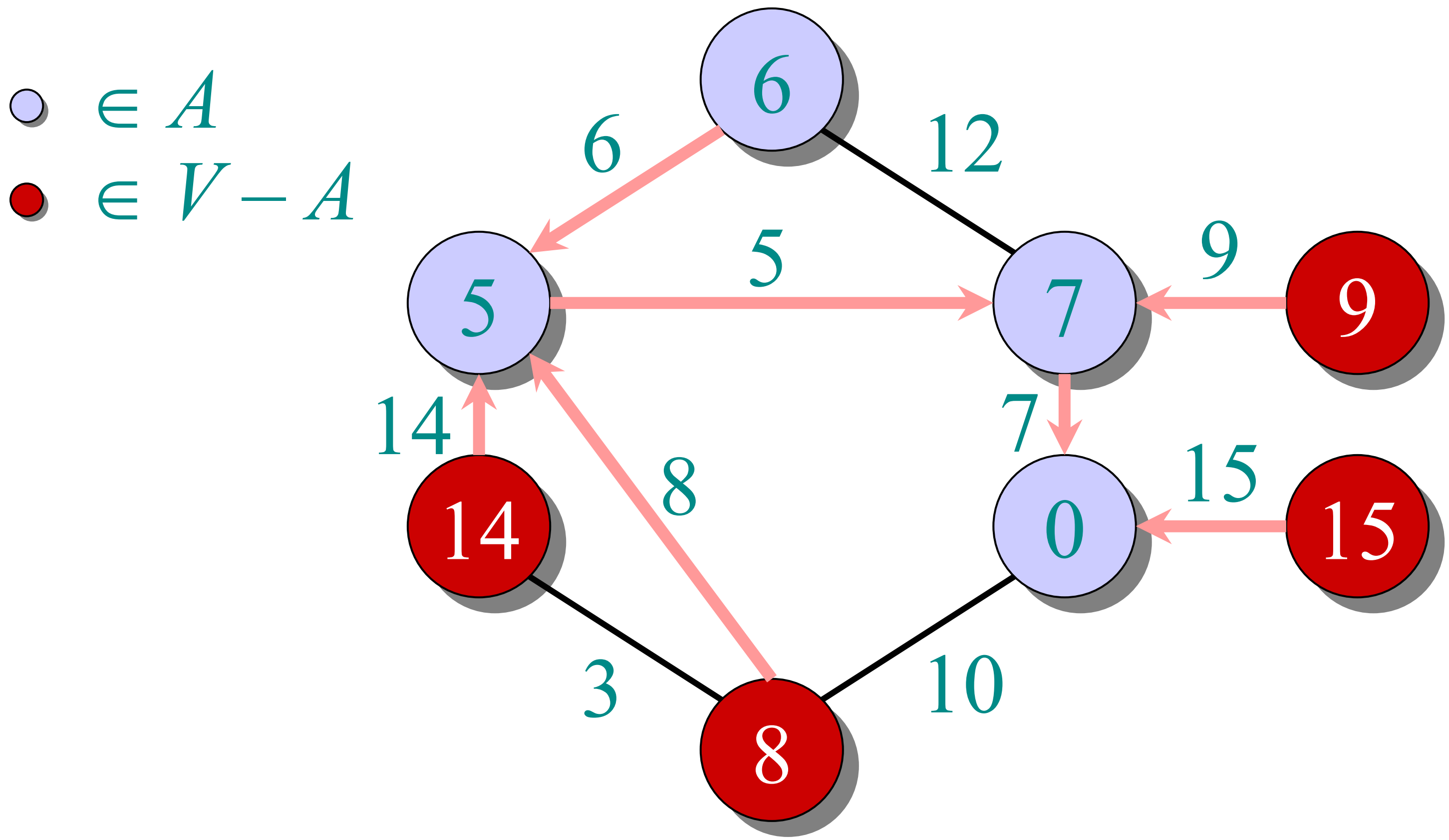


Example of Prim's algorithm



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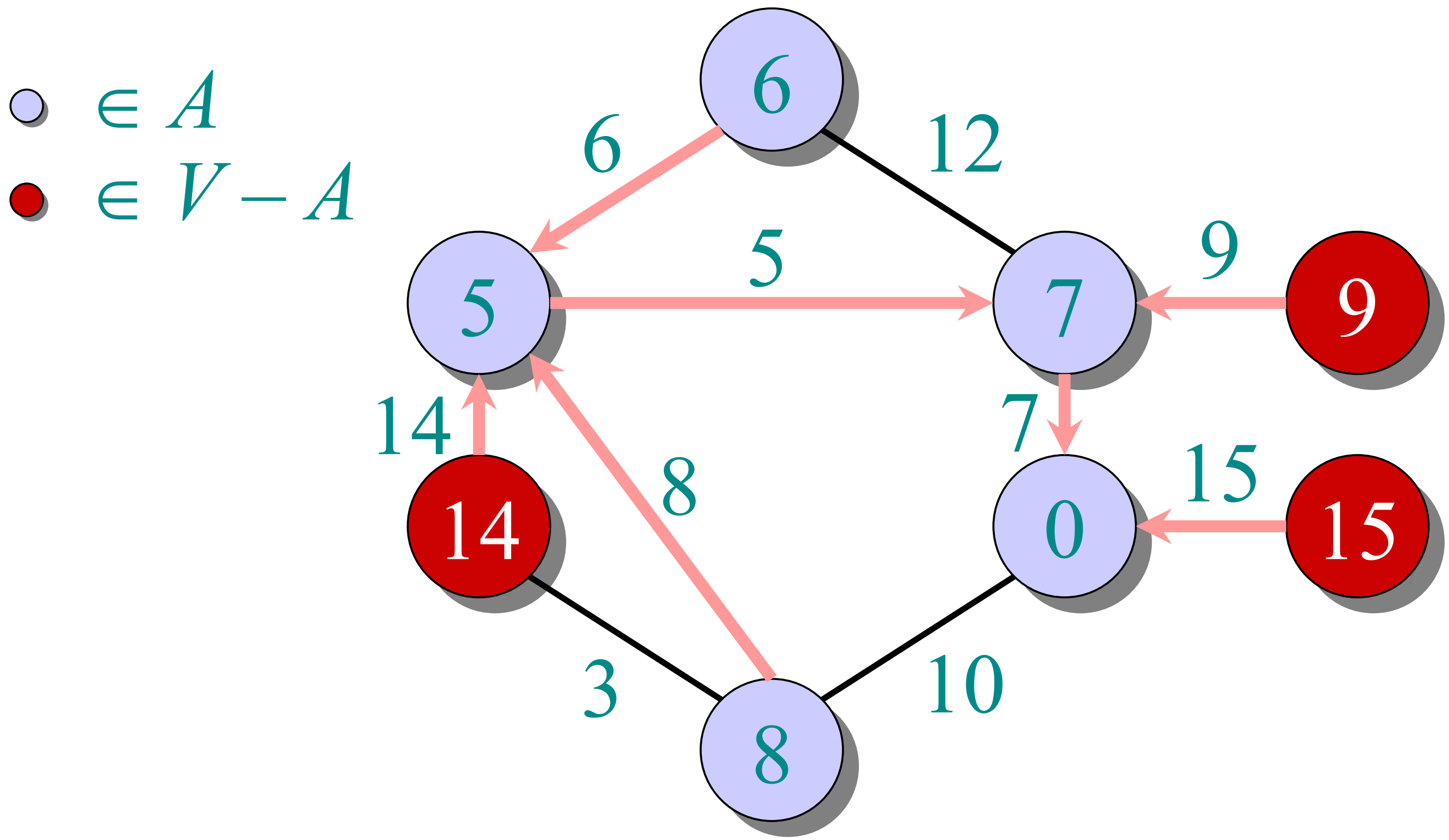

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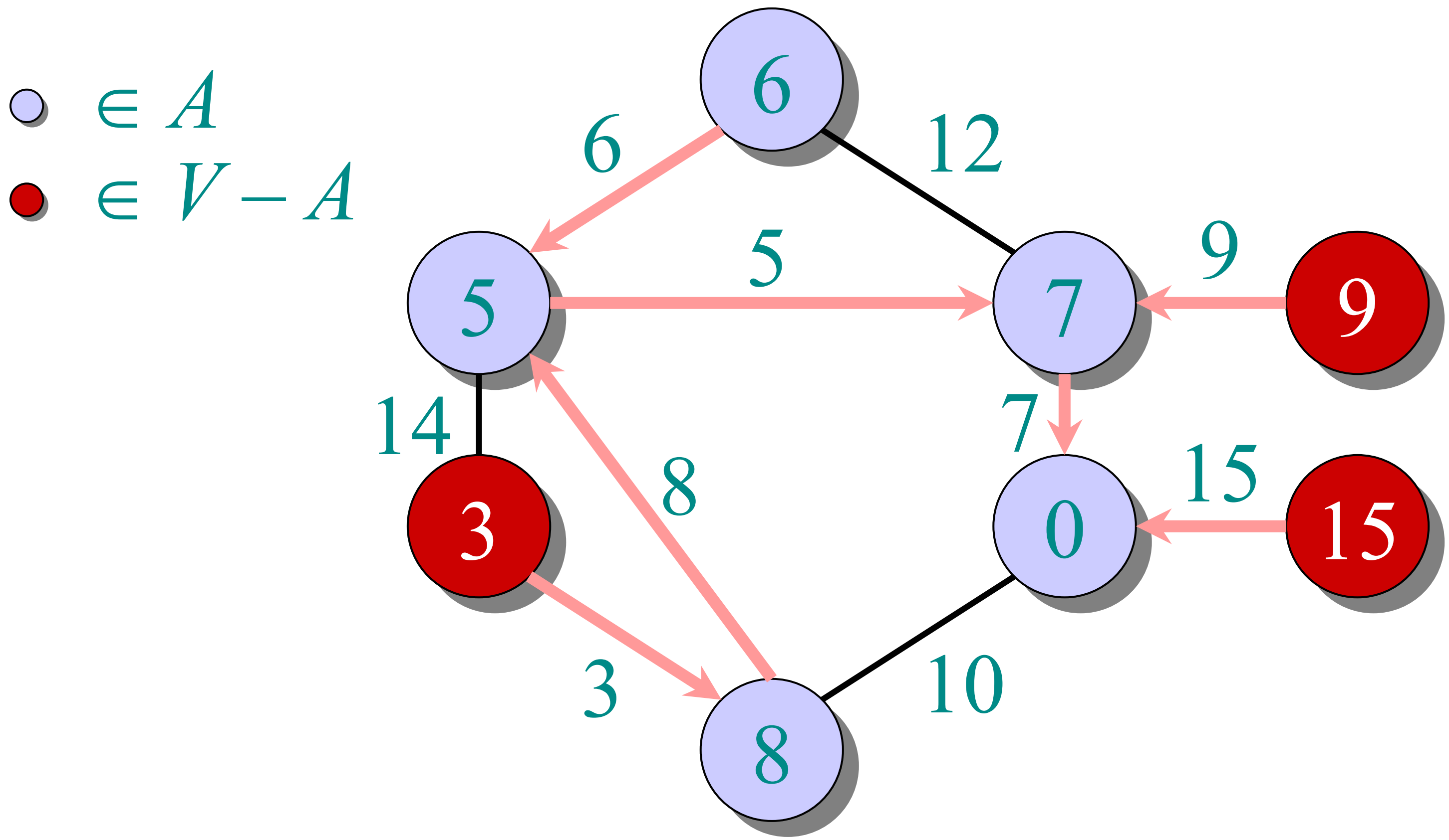

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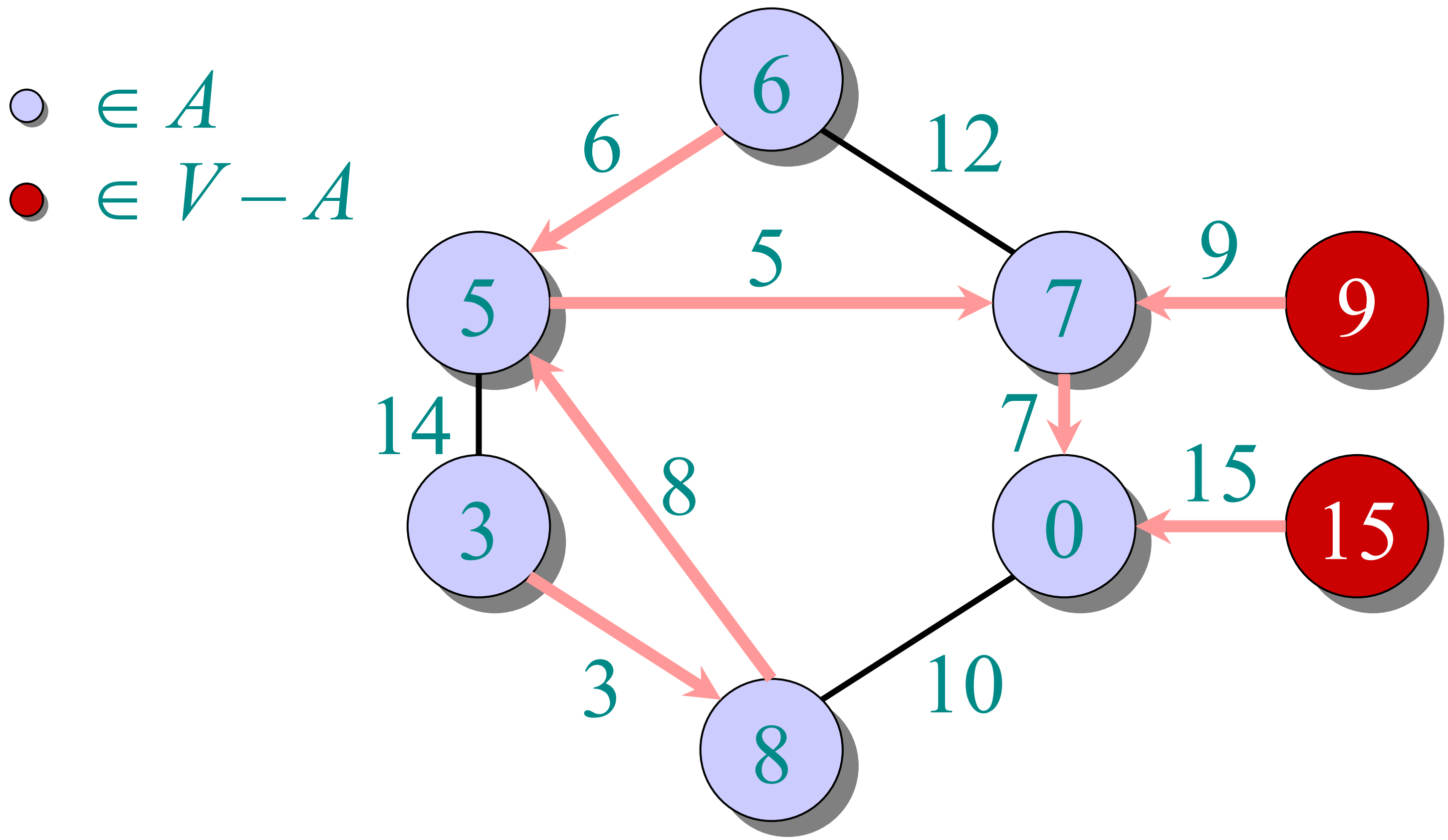

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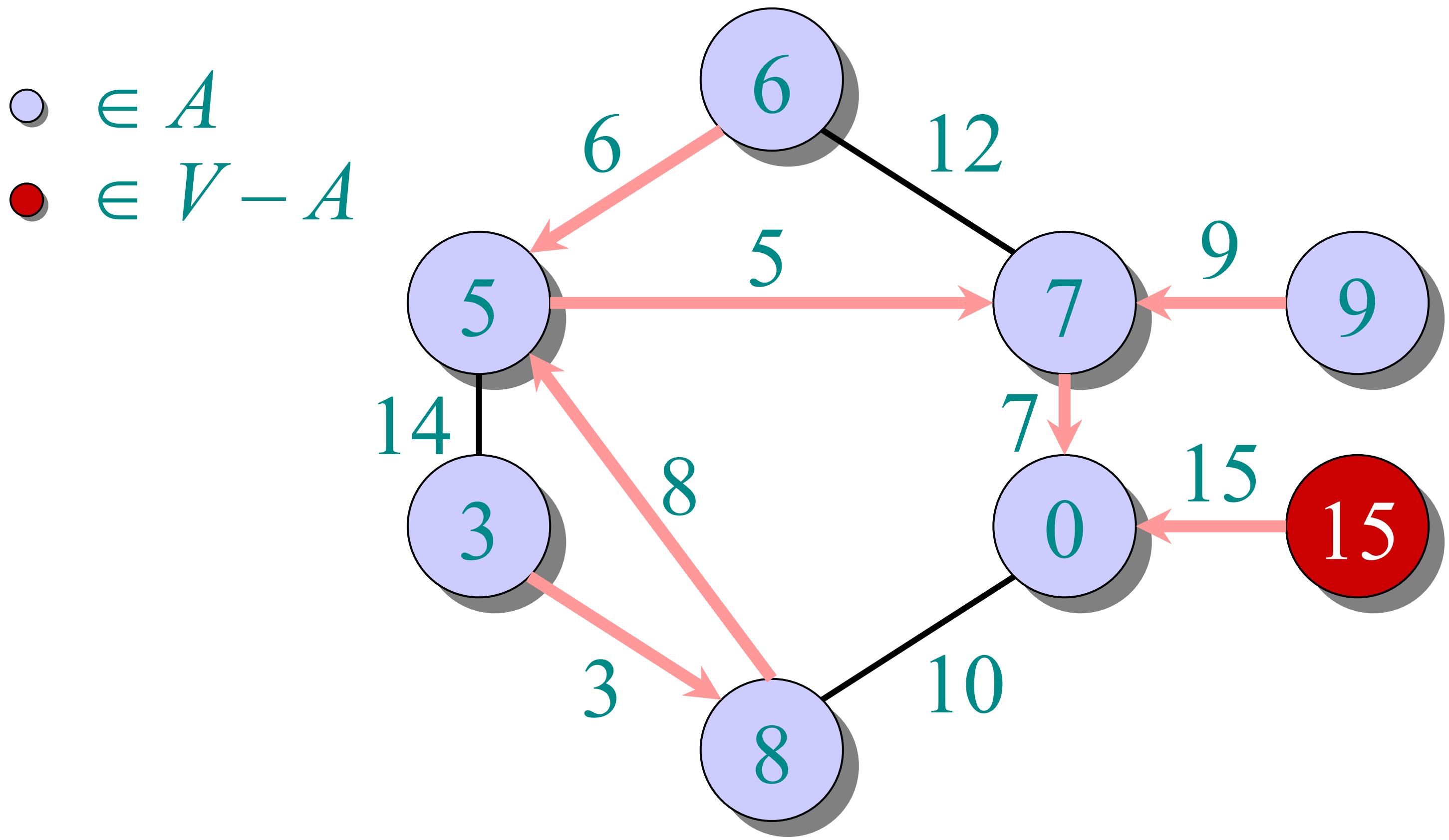
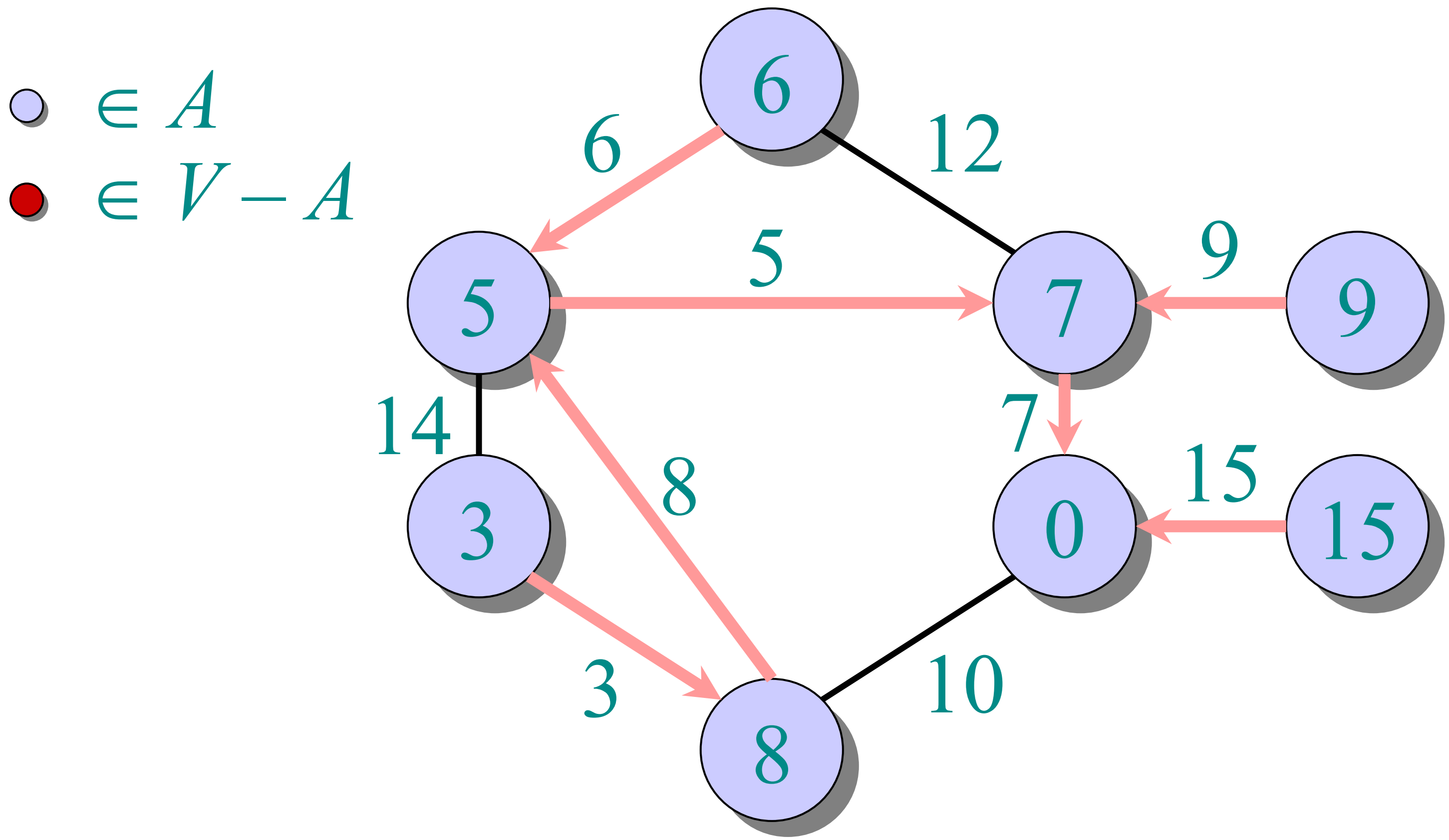


Example of Prim's algorithm



Analysis of Prim

$\Theta(V)$ total $\left\{\begin{array}{l} Q \leftarrow V \\ key[v] \leftarrow \infty \text{ for all } v \in V \\ key[s] \leftarrow 0 \text{ for some arbitrary } s \in V \end{array}\right.$

while $Q \neq \emptyset$

$|V|$ times, with $degree(u)$ times for the inner loop:

$$
\begin{array}{l}
\textbf{while } Q \neq \emptyset \\
\quad \textbf{do } u \leftarrow \text{EXTRACT-MIN}(Q) \\
\qquad \textbf{for each } v \in Adj[u] \\
\qquad\quad \textbf{do if } v \in Q \text{ and } w(u, v) < key[v] \\
\qquad\qquad \textbf{then } \boxed{key[v] \leftarrow w(u, v)} \\
\qquad\qquad\qquad \pi[v] \leftarrow u
\end{array}
$$

Handshaking Lemma $\Rightarrow$ $\Theta(E)$ implicit DECREASE-KEY's.

Time $= \Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$

Analysis of Prim (continued)

Time = $\Theta(V)\cdot T_{\text{EXTRACT-MIN}} + \Theta(E)\cdot T_{\text{DECREASE-KEY}}$

Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
array	$O(V)$	$O(1)$	$O(V^2)$
binary heap	$O(\lg V)$	$O(\lg V)$	$O(E \lg V)$
Fibonacci heap	$O(\lg V)$ amortized	$O(1)$ amortized	$O(E + V \lg V)$ worst case

MST algorithms

Kruskal's algorithm (see CLRS):
- Uses the ***disjoint-set data structure*** (Lecture 20).
- Running time $= O(E \lg V)$.

Best to date:
- Karger, Klein, and Tarjan [1993].
- Randomized algorithm.
- $O(V + E)$ expected time.