ARITHMETIC FUNCTIONS

PERFECT, AMICABLE, SOCIABLE NUMBER

DEFINITION

Let n be a positive integer.
Let $(x_1,x_2,\ldots,x_k)$ be k positive integers,
all greater than 1, satisfying:

$\sigma(x_1) = x_1 + x_2.$

$\sigma(x_2) = x_2 + x_3.$

…

$\sigma(x_k) = x_k + x_1.$

Then the k positive integers form a sociable group with order k (k-cycle).

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If k = 1, $\sigma(x_1) = x_1 + x_1 = 2x_1$,
x_1 is called perfect number.

If k = 2, $\sigma(x_1) = x_1 + x_2 = \sigma(x_2)$,
(x_1,x_2) is called an amicable pair.

The k integers, $x_1,x_2,\ldots,x_k$,

$\sigma(x_1) = \sigma(x_2) = \ldots = \sigma(x_k) = x_1+x_2+\ldots+x_k$,
integers $x_1,x_2,\ldots,x_k$ are called an amicable k-tuple.

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EXAMPLES

the first four perfect numbers:
6, 28, 496, 8128

The first three amicable pairs:
(220,284), (1184,1210), (2620,2924), (5020,5564)

The first three amicable triples:
(1980,2016,2556), (9180,9504,11556), (21668,22200,27312)

etc…

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THEOREM
(THE EUCLID-EULER THEOREM)

Let n be an even perfect number
if and only if
$n = 2^{p-1} (2^p - 1)$,
where $2^p - 1$ is a Mersenne prime.

THEOREM
(THABIT's RULE FOR AMICABLE PAIRS)

Let $p = 3 \times 2^{n-1} - 1$.
Let $q = 3 \times 2^n - 1$.
Let $r = 9 \times 2^{2n-1} - 1$.
If p, q and r are primes,
then $(2^n \times p \times q, 2^n \times r)$ is amicable pair.

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THEOREM
(EULER'S RULE FOR AMICABLE PAIRS)

Let n be a positive number,
and choose $0 < x < n$ such that
$g = 2^{n-x} + 1$.
if

$p = (2^x \times g) - 1$,
$q = (2^n \times g) - 1$,
$s = (2^{n+x} \times g^2) - 1$,

are primes,
then $(2^n \times p \times q,\ 2^n \times s)$ is amicable pair.

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PERFECT, AMICABLE, SOCIABLE NUMBER

THEOREM

Given an amicable pair $(M,N) = (a \times u,\ a \times p)$,
with gvd(a,u) = gcd(a,p) = 1, and p is prime.

If a pair of primes (r, s) with $p < r < s$,
and $\gcd(a, r \times s) = 1$, exists,
satisfying the bilinear Diophantine equation

$(r - p)(s - p) = (\sigma(a) \times \sigma(u)^2)/a$ and
and a third prime q exists, with $\gcd(a \times u, q) = 1$
and $q = r + s + u$.

then $(a \times u \times q,\ a \times r \times s)$ is amicable pair.

ARITHMETIC FUNCTIONS

ϕ-function (Euler's)

DEFINITION

Let n be a positive integer.
Euler's (totient) ϕ-function,
$\phi(n)$ is defined to be the number of positive integers $k < n$ which are relatively prime to n:

$$\phi(n) = \sum_{(1 \le k < n,\ \gcd(k,n) = 1)} 1.$$

EXAMPLE

n	$\phi(n)$	n	$\phi(n)$	n	$\phi(n)$
1	1	5	4	9	6
2	1	6	2	10	4
3	2	7	6	100	40
4	2	8	4	101	100

ARITHMETIC FUNCTIONS

ϕ-function (Euler's)

THEOREM

Let n be a positive integer. Then

$\phi(n)$ is multiplicative. i.e., $\phi(mn) = \phi(m)\phi(n)$.

If p is a prime, then $\phi(p) = p - 1$.
More generally, if n is a prime power, p^u,
then $\phi(p^u) = p^u - p^{u-1}$.

If n is a composite, then
$\phi(n) = p_1^{u1}(1-(1/p_1))p_2^{u2}(1-(1/p_2)) \times \ldots \times p_k^{uk}(1-(1/p_k))$
with $n = p_1^{u1}p_2^{u2}\ldots p_k^{uk}$ (prime factorization form).

ARITHMETIC FUNCTIONS

λ-function (Carmichael's)

DEFINITION

Let n be a positive integer.
Carmichael's λ-function, λ(n) is defined as follows

$\lambda(n) = \phi(n)$ if n is a prime.

$\lambda(p^u) = \phi(p^u)$ for $p = 2$ and $u \le 2$ and for $p \ge 3$.

$\lambda(2^u) = \phi(2^u)/2$ for $u \ge 3$

$\lambda(n) = \text{lcm}(\lambda(p_1^{u1}), \lambda(p2u2), \ldots, \lambda(p_k^{uk}))$,
with $n = p_1^{u1} p_2^{u2} \ldots p_k^{uk}$ (prime factorization form).

ARITHMETIC FUNCTIONS

λ-function (Carmichael's)

EXAMPLE

n	λ(n)	φ(n)
1	1	1
2	1	1
3	2	2
4	2	2
5	4	4
6	2	2
7	6	6
8	2	4
9	6	6
10	4	4
100	20	40
101	100	100
102	16	32

ARITHMETIC FUNCTIONS

μ-function (Mobius)

DEFINITION

Let n be a positive integer.
Mobius μ-function, μ(n) is defined as follows

$\mu(n)$	$= 1$	if $n = 1$,
	$= 0$	if n contains a squared factor,
	$= (-1)^k$	if $n = p_1p_2\ldots p_k$.

ARITHMETIC FUNCTIONS

μ-function (Mobius)

EXAMPLE

n	$\lambda(n)$	$\phi(n)$	$\mu(n)$
1	1	1	1
2	1	1	-1
3	2	2	-1
4	2	2	0
5	4	4	-1
6	2	2	1
7	6	6	-1
8	2	4	0
9	6	6	0
10	4	4	1
100	20	40	0
101	100	100	-1
102	16	32	-1

ARITHMETIC FUNCTIONS

μ-function (Mobius)

THEOREM

Let n be a positive integer. Then

$\mu(n)$ is multiplicative. i.e., $\mu(mn) = \mu(m)\,\mu(n)$.

Let $\nu(n) = \Sigma_{(d|n)}\,\mu(d)$. Then
$\nu(n) = 1$ if $n = 1$,
$= 0$ if $n > 1$.

ARITHMETIC FUNCTIONS

μ-function (Mobius)

THEOREM

THE MOBIUS INVERSION FORMULA
If f is any arithmetic function and
if $g(n) = \Sigma_{(d|n)}\, f(d)$,
$\mu(n) = \Sigma_{(d|n)}\,\mu(n/d)\, g(d) = \Sigma_{(d|n)}\,\mu(d)\, g(n/d)$.

THE CONVERSE OF THE MOBIUS INVERSION FORMULA
If $f(n) = \Sigma_{(d|n)}\,\mu(n/d)\, g(d)$,
then $g(n) = \Sigma_{(d|n)}\, f(d)$.

$$\phi(n) = \Sigma_{(d|n)}\,(\,\mu(d)/d\,)$$