2110355 FORMAL LANGUAGES AND AUTOMATA THEORY

INTRODUCTION
LOGIC SET RELATION FUNCTION

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DESCRIPTION

Studies concepts of grammars, automata, languages, computability and complexity ; the relationship between automata and various classes of languages; Turing machine and equivalent models of computation, the Chomsky hierarchy, context-free grammar, push-down automata, etc.; pumping lemmas and variants, closure properties and decision properties; parsing algorithms.

EVALUATION

◆ Homework	15 %
◆ Quiz	20 %
◆ Mid-Term examination	30 %
◆ Final examination	35 %

REFERENCES

- Introduction to Languages and Theory of Computation(2nd ed.) John C. Martin
- Introduction to Computer Theory (2nd ed.) Daniel I. A. Cohen
- Discrete Mathematics and its Applications (4th ed.) Kenneth H. Rosen
- Languages and Machines: An Introduction to the Theory of Computer Science (2nd ed.) Thomas A. Sudkamp

BACKGROUND

Charles Babbage

1791-1871

- Created the first difference engine (producing the members of the sequence $n^2 + n + 41$ at the rate of about 60 every 5 minutes)
- The 1st drawings of the analytical engine (describes five logical components, the store, the mill, the control, the input and the output)

- The construction of modern computers, logically similar to Babbage's design

Kurt Gödel

1906-1978

Proved that there was no algorithm to provide proofs for all the true statements in mathematics.

Universal model for all algorithms.

VARIOUS VERSIONS

OF A UNIVERSAL ALGORITHM MACHINE

◆ Andrei Andreevich Markov	1856-1922
◆ Emil Post	1897-1954
◆ Alonzo Church	1903-1995
◆ Stephen Kleene	1909-1994
◆ John von Neumann	1903-1957
◆ Alain Turing	1912-1954

Alain Turing

1912-1954

Computing machinery and intelligence

•studied problems which today lie at the heart of artificial intelligence.
•proposed the Turing Test which is still today the test people apply in attempting to answer whether a computer can be intelligent.

Warren McCulloch & Walter Pitts

neurophysiologists

Constructed for a “neural net” was a theoretical machine of the same nature as the one Turing invented.

Modern linguists

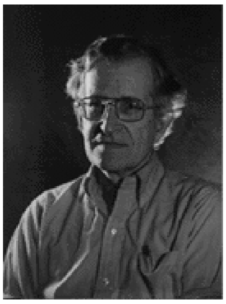
Investigated a very similar subject

•What is language in general ?
•How could primitive humans have developed language ?
•How do people understand it ?
•How do they learn it as children ?
•What ideas can be expressed, and in what way ?
•How do people construct sentences from the idea s in their minds ?

Noam Chomsky

Massachusetts Institute of Technology

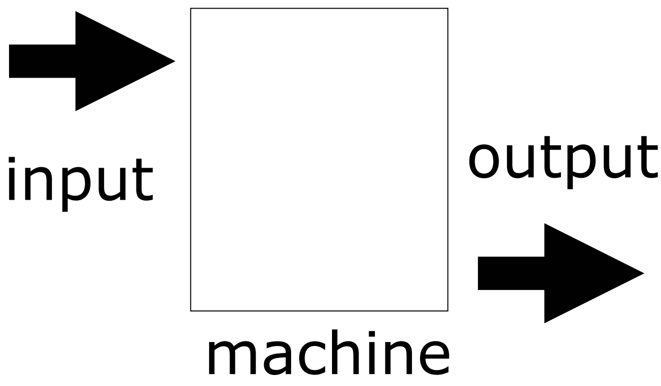
Created the subject of mathematical models for the description of languages to answer these questions.



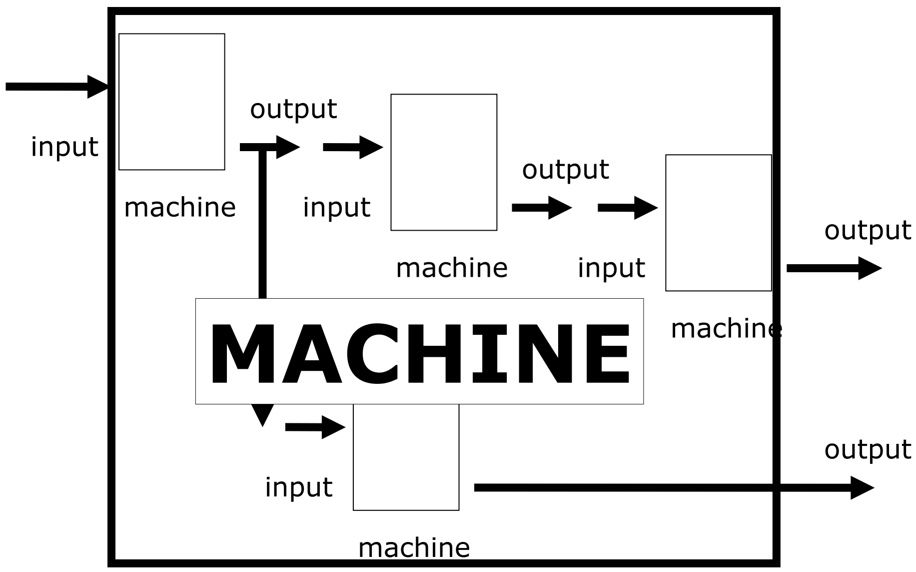
MAIN TOPIC

We shall study different types of theoretical machines that are mathematical models for actual physical processes.

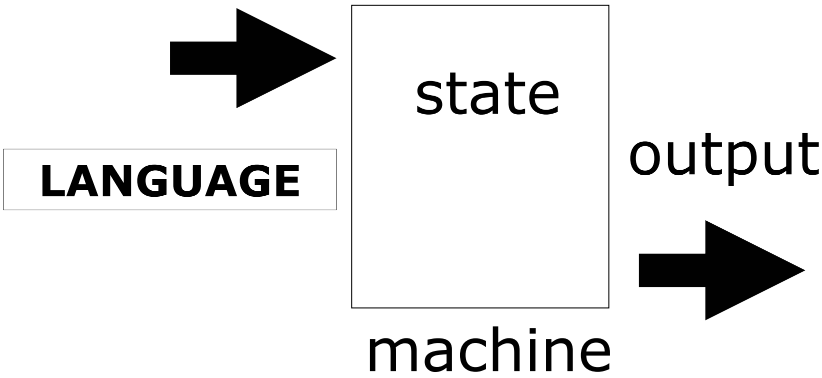
MACHINE MODEL



MACHINE MODEL



MACHINE MODEL



MACHINE MODEL

?

MAIN CONCLUSIONS

" this can be done or it can never be done."

AGENDA

- Regular languages
  - DETERMINISTIC STATE MACHINES
  - TRANSITION GRAPHS
  - NONDETERMINISTIC STATE MACHINES
- Context-Free languages
  - PUSH-DOWN AUTOMATA
  - TURING MACHINES
- Context-sensitive languages
- Recursive languages
- Recursively enumerable languages

BACKGROUND KHOWLEDGE

- Logic
- Set Relation & function
- Theory & methods of proof
- Asymptotic notation

LOGIC

SYLLOGISTIC REASONING

Aristotle (384-322 B.C.)

Organon: the first treatise on logic.

The fundamental elements of this logic are *terms* and *arguments* are evaluated as good or bad depending on how the terms are arranged in the argument.

SYLLOGISTIC REASONING
Aristotle (384-322 B.C.)

There are four different types of Syllogistic arguments used to describe things with logic.

- All A are B (universal affirmative)
- No A are B (universal negative)
- Some A are B (particular affirmative)
- Some A are not B (particular negative)

SYLLOGISTIC REASONING
Aristotle (384-322 B.C.)

There are four different types of Syllogistic arguments used to describe things with logic.

- All cats are animals.
- No A are B (universal negative)
- Some A are B (particular affirmative)
- Some A are not B (particular negative)

SYLLOGISTIC REASONING

Aristotle (384-322 B.C.)

There are four different types of Syllogistic arguments used to describe things with logic.

- All cats are animals.
- No cats are plants.
- Some A are B (particular affirmative)
- Some A are not B (particular negative)

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- Some animals are cats.
- Some A are not B (particular negative)

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- No cats are plants.
- Some animals are cats.
- Some animals are not cats.

SYLLOGISTIC REASONING

Aristotle (384-322 B.C.)

In fact in Prior Analytics Aristotle proposed the now famous Aristotelian syllogistic, a form of argument consisting of two premises and a conclusion.

His example is:-

- Every Greek is a person
- Every person is mortal
- Every Greek is mortal

This does not facilitate large compound formulas, however.

DEDUCTIVE REASONING

Euclid of Alexandria (325-265 B.C.)

Thirteen Books of Elements:

Axiom-Definition-Theorem-Proof

- BOOK VII-Proposition2-3: Euclid's algorithm for computing the greatest common divisor
- BOOK IX-Proposition20: infinitely many primes
- BOOK IX-Proposition21-29: Properties of parity of integers
- BOOK IX-Proposition36: Perfect numbers

DEDUCTIVE REASONING

Euclid of Alexandria (325-265 B.C.)

Thirteen Books of Elements:

Axiom-Definition-Theorem-Proof

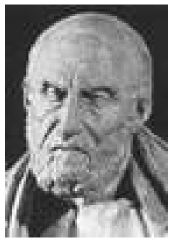
The style of Euclid's work has become the standard for formal mathematical writing up to the present day.

MODAL LOGIC

Chrysippus of Soli (279-206 B.C.)

Developed a logic based on whole propositions

- Every proposition is either true or false.
- The truth of compound propositions depends on the truth or falsity of the component parts.
- The foundations for the truth-functional account of logic
- Modal logic



COMPOUND LOGIC

Claudius Galenus: Galen of Pergamum (129-199 A.D.)

Developed a theory of compound categorical syllogis

- Greek doctor
- Introducing the idea of opposites to treat illnesses

ITALIAN PHILOSOPHER

Ancius Manlius Severinus Boethius (480-524 A.D.)

Translated and wrote commentaries on the work of Aristotle and Chrysippus.

SYMBOLIC LOGIC

Gottfried Wilhelm von Leibniz (1646-1716 A.D.)

Invented the first artificial language for logic.

The fundamental theorem of his metaphysics of concepts

PROPOSITIONAL LOGIC

George Boole (1815-1864 A.D.)

Invented Boolean algebra

- The Mathematical Analysis of Logic
- The Laws of Thought

PROPOSITIONAL LOGIC

George Boole (1815-1864 A.D.)

A **proposition** is a statement that is either *true* or *false*, but not both.

Example: propositions

Every Greek is person.	p
Every person is mortal.	q
Every Greek is immortal.	r

Propositions p and q are true, but r is false.

PROPOSITIONAL LOGIC

Augustus De Morgan (1806-1871 A.D.)

Invented & traduced the term of "mathematical induction"

- De Morgan's article
  Induction (Mathematics) in the *Penny Cyclopedia* (1838)
- introduced De Morgan's laws
  and his greatest contribution is as a reformer of mathematical logic.

PROPOSITIONAL LOGIC

Augustus De Morgan (1806-1871 A.D.)

Invented & traduced the term of "mathematical induction"

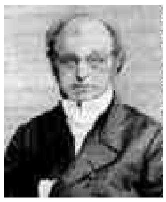
EXAMPLE

It is raining today.

It is cold today.

It is not the case that it is raining or is cold today.

Today it is not cold and it is not raining.



PROPOSITIONAL LOGIC

- Identifying logical form
- Statements
- Compound statements
- Negation, Conjunction & Disjunction
- Exclusive or
- Applications
- Conditional statements
- Exercises

Logical ForM

IDENTIFYING LOGICAL FORM

Example

If "Automata" is easy or I study hard, then I will get an A in this course.

Illustrate this sentence by a logical form which alphabet is usually used to represent their components.

p **denotes** *"Automata" is easy.*

q **denotes** *I study hard.*

r **denotes** *I will get an A in this course.*

If p **or** q**, then** r

Logical ForM
PROPOSITION

Definition

A proposition is a statement that is either true or false but not both.

Examples

Budapest is the capital of Romania.
Five plus four equals nine.

FLASE

TRUE

Example *He is a student.*

The truth or falsity of this statement depends on "he".

TRUTH VALUE

Logical ForM
NEGATION

Definition

Let p be a proposition. The statement

"It is not the case that p."

is another proposition, called the negation of p.
The negation of p is denoted by $\neg p$, read "not p".

Example

p **denotes "***Today is Sunday***".**

FLASE

TRUE

Then,

"*It is not the case that today is Sunday*" **is denoted by** $\neg p$

The truth or falsity of the 2nd statement depends on the first one.

OPPOSITE TRUTH VALUE

TRUTH VALUE

Logical ForM
NEGATION

DEFINITION

The negation of p has opposite truth value from p.

TRUTH TABLE

p	$\neg p$
T	F
F	T

Logical ForM
CONJUNCTION

Definition

Let p and q be propositions. The compound proposition

"p and q" *(conjunction of p and q)*

denoted by $p \wedge q$, is the statement that is true when both p and q are true and is false otherwise.

Example

p **denotes "***Today is Sunday***".** **FALSE**

q **denotes "***John goes to school***".** **TRUE**

Then,

"*Today is Sunday and John goes to school*" **is denoted by $p \wedge q$.** **FALSE**

TRUTH VALUE

Logical ForM
CONJUNCTION

DEFINITION

The conjunction of p and q, is true when, and only when, both p and q are true.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

2^n

Logical ForM
DISJUNCTION

Definition

Let p and q be propositions. The compound proposition

"p **or** q" *(disjunction of p and q)*

denoted by $p \vee q$, is the statement that is false when both p and q are false and is true otherwise.

Example

p **denotes** "*Diana goes to school*". **FALSE**

q **denotes** "*John goes to school*". **TRUE**

Then,

"*Diana goes to school or John goes to school*" **is denoted by** $p \vee q$. **TRUE**

TRUTH VALUE

Logical ForM
DISJUNCTION

DEFINITION

The disjunction of p and q, is true when at least one of p or q is true.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Logical ForM
MORE GENERAL COMPOUND STATEMENTS

Logical connectives (ordered)

- **Negation**
- **Conjunction, Disjunction**

EXAMPLE

(1) It is raining but it is not cold.
(2) It is neither raining nor cold.

Let p = "It is raining",
q = "It is cold".

(1) $p \wedge \neg q$

(2) $\neg p \wedge \neg q$

TRUTH VALUE

Logical ForM
MORE GENERAL COMPOUND STATEMENTS

DEFINITION

A propositional form is an expression made up of propositions (p,q,...) and logical connectives.

A propositional form is also a proposition.

EXAMPLE

$$(p \wedge q) \vee \neg p$$

p	q	$p \wedge q$	$\neg p$	$(p \wedge q) \vee \neg p$
T	T	T	F	T
T	F	F	F	F
F	T	F	T	T
F	F	F	T	T

Logical ForM
MORE GENERAL COMPOUND STATEMENTS

EXAMPLE

A proposition is either true or false, but not both.

Let p = "A proposition is true",

q = "A proposition is false".

1. $(p \wedge \neg q) \vee (\neg p \wedge q)$

2. $(p \vee q) \wedge \neg(p \wedge q)$

TRUTH VALUE

Logical ForM
EXCLUSIVE OR

DEFINITION

Exclusive or, denoted by ⊕, means "or but not both".

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

TRUTH VALUE

Logical ForM
EXCLUSIVE OR

DEFINITION

Exclusive or, denoted by ⊕, means "or but not both".

TRANSLATE $p \oplus q$ **INTO SYMBOL** $(p \vee q) \wedge \neg(p \wedge q)$.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$p \vee q$	$(p \vee q) \wedge \neg(p \wedge q)$
T	T	T	F	T	F
T	F	F	T	T	T
F	T	F	T	T	T
F	F	F	T	F	F

$p \oplus q$
F
T
T
F

TRUTH VALUE

Logical ForM
EXCLUSIVE OR

DEFINITION

Exclusive or, denoted by $\oplus$, means "or but not both".

TRANSLATE $p \oplus q$ **INTO SYMBOL** $(p \vee q) \wedge \neg(p \wedge q)$.

DEFINITION

Two propositions are called logically equivalent,
Usually denoted by $\Leftrightarrow$,
If, and only if, they have identical truth values
for each possible substitution of propositions
for their statement variables.

EXAMPLE: $p \oplus q \Leftrightarrow (p \vee q) \wedge \neg(p \wedge q)$.

TRUTH VALUE

Logical ForM
LOGICALLY EQUIVALENT

P	Q	P $\Leftrightarrow$ Q
T	T	T
T	F	F
F	T	F
F	F	T

DEFINITION

Two propositions are called logically equivalent,
Usually denoted by $\Leftrightarrow$,
If, and only if, they have identical truth values
for each possible substitution of propositions
for their statement variables.

EXAMPLE: $p \oplus q \Leftrightarrow (p \vee q) \wedge \neg(p \wedge q)$.

TRUTH VALUE

Logical ForM

TAUTOLOGY

DEFINITION

A tautology is a propositional form that is always true.

EXAMPLE: $p \oplus q \Leftrightarrow (p \vee q) \wedge \neg(p \wedge q).$

DEFINITION

A contradiction is a propositional form that is always false.

DEFINITION

A propositional form that is neither tautology nor contradiction is called a contingency.

Application on circuitS

COMPUTER ADDITION

Consider the question of designing a circuit to produce the sum of two binary digits p and q.

p		q		carry	sum
1	+	1	=	1	0
1	+	0	=	0	1
0	+	1	=	0	1
0	+	0	=	0	0

$1 \Leftrightarrow$ TRUE

$0 \Leftrightarrow$ FALSE

p	q	Carry	Sum
1	1	1	0
1	0	0	1
0	1	0	1
0	0	0	0

Application on circuitS
COMPUTER ADDITION

Consider the question of designing a circuit to produce the sum of two binary digits p and q.

p		q		carry	sum
1	+	1	=	1	0
1	+	0	=	0	1
0	+	1	=	0	1
0	+	0	=	0	0

$1 \Leftrightarrow$ TRUE

$0 \Leftrightarrow$ FALSE

p	q	Carry $\Leftrightarrow p \wedge q$	Sum $\Leftrightarrow p \oplus q$
1	1	1	0
1	0	0	1
0	1	0	1
0	0	0	0

Application on circuitS
COMPUTER ADDITION

Consider the question of designing a circuit to produce the sum of two binary digits p and q.

p	q	Carry $\Leftrightarrow p \wedge q$	Sum $\Leftrightarrow p \oplus q$

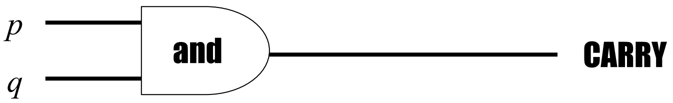
CARRY $\Leftrightarrow$ (p and q) SUM $\Leftrightarrow$ (p or q) and (not (p and q))

Application on circuit**S**
COMPUTER ADDITION

Consider the question of designing a circuit to produce the sum of two binary digits p and q.

p	q	Carry $\Leftrightarrow p \wedge q$	Sum $\Leftrightarrow p \oplus q$

CARRY $\Leftrightarrow$ **(** p **and** q **)**   **SUM** $\Leftrightarrow$ **(** p **or** q **) and (not (** p **and** q **))**

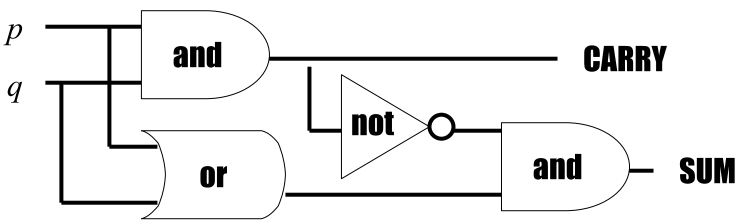


Application on circuit**S**
COMPUTER ADDITION

Consider the question of designing a circuit to produce the sum of two binary digits p and q.

p	q	Carry $\Leftrightarrow p \wedge q$	Sum $\Leftrightarrow p \oplus q$

CARRY $\Leftrightarrow$ **(** p **and** q **)**   **SUM** $\Leftrightarrow$ **(** p **or** q **) and (not (** p **and** q **))**



SummarY

Theorem : Logical Equivalences,
given any propositions p,q and r, a tautology T and a contradiction C, the following logical equivalences hold:

- **Commutative laws:** $p \wedge q \Leftrightarrow q \wedge p$ $p \vee q \Leftrightarrow q \vee p$
- **Associative laws:** $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$
  $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
- **Distributive laws:** $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
  $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
- **Identity laws:** $p \wedge \mathrm{T} \Leftrightarrow p$ $p \vee \mathrm{C} \Leftrightarrow p$
- **Domination laws:(Universal bound laws)** $p \vee \mathrm{T} \Leftrightarrow \mathrm{T}$ $p \wedge \mathrm{C} \Leftrightarrow \mathrm{C}$
- **Idempotent laws:** $p \wedge p \Leftrightarrow p$ $p \vee p \Leftrightarrow p$
- **Negation laws:** $p \vee \neg p \Leftrightarrow \mathrm{T}$ $p \wedge \neg p \Leftrightarrow \mathrm{C}$
- **Double negative laws:** $\neg(\neg p) \Leftrightarrow p$
- **De Morgan's laws:** $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$
  $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$
- **Absorption laws:** $p \vee (p \wedge q) \Leftrightarrow p$ $p \wedge (p \vee q) \Leftrightarrow p$

ExampleS

1. Show that $(p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge q) \Leftrightarrow (p \vee q)$ is a tautology.
2. Show that $\neg(\neg p \wedge (p \vee q)) \vee q$ is a tautology.
3. Verify the distributive law:
   $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$.

Conditional StatementS
IMPLICATION

Let p, q be propositions. A statement of the form

If p then q, denoted by $p \to q$

where p is called *hypothesis*

q is called *conclusion*.

The statement is false when p is true and q is false.

Example

If 3.201 is divisible by 6, then 3.201 is divisible by 3.

The truth value of this sentence is **TRUE**

Conditional StatementS
IMPLICATION

DEFINITION

Let p, q be propositions.
The conditional of q by p is "If p then q" or "p implies q" and is denoted by $p \to q$.
It is false when p is true and q is false, and true otherwise.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Conditional StatementS
IMPLICATION

EXAMPLE

Construct a truth table of $(p \vee \neg q) \to \neg p$.

p	q	$\neg p$	$\neg q$	$p \vee \neg q$	$(p \vee \neg q) \to \neg p$
T	T	F	F	T	F
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Conditional StatementS
CONTRAPOSITIVE

DEFINITION

The contrapositive of a conditional statement of the form "If p then q" **is**

"If $\neg q$ then $\neg p$".

Symbolically, it is $\neg q \to \neg p$.

p	q	$\neg q \to \neg p$
T	T	T
T	F	F
F	T	T
F	F	T

A conditional statement is logically equivalent to its contrapositive.

Conditional StatementS

CONTRAPOSITIVE

DEFINITION

The contrapositive of a conditional statement of the form
"If p then q" **is**

"If $\neg q$ then $\neg p$".

Symbolically, it is $\neg q \rightarrow \neg p$.

EXAMPLE

A conditional statement
IF today is Monday, tomorrow is Tuesday.
The contrapositive
IF tomorrow is not Tuesday, today is not Monday.

Conditional StatementS

- **CONVERSE**
  The converse of $p \rightarrow q$ **is** $q \rightarrow p$.
- **INVERSE**
  The inverse of $p \rightarrow q$ **is** $\neg p \rightarrow \neg q$.
- **ONLY IF**
  p **only if** q **means If** $\neg q$ **then** $\neg p$.

EXAMPLE

STATEMENT
If John can swim across the lake, he can swim to the island.
CONVERSE
If John can swim to the island, he can swim across the lake.

Conditional Statements

- **CONVERSE**

  The converse of $p \to q$ is $q \to p$.
- **INVERSE**

  The inverse of $p \to q$ is $\neg p \to \neg q$.
- **ONLY IF**

  p only if q means If $\neg q$ then $\neg p$.

EXAMPLE

STATEMENT

If John can swim across the lake, he can swim to the island.

INVERSE

If John cannot swim across the lake, he cannot swim to the island.

Conditional Statements

- **ONLY IF**
- p only if q means If $\neg q$ then $\neg p$.

EXAMPLE

ONLY IF-STATEMENT

John will break the world's record for the mile run only if
He runs the mule in under four minutes.

MEANING

- IF John does not run the mile in under four minutes,
  then he will not break the world's record.
- If John breaks the world's record,
  then he will have run the mile in under four minutes.

~~John will break the world's record,~~
~~if he runs the mile in under four minutes.~~

Conditional StatementS
NECESSARY & SUFFICIENT

DEFINITION
Given p and q are statements.
p is a sufficient condition for q means

"If p then q".

p is a necessary condition for q means

"If $\neg p$ then $\neg q$"
or "If q then p".

CONSEQUENTLY,
p is a necessary and sufficient condition for q means

"p if, and only if, q"

$p \leftrightarrow q$.

BICONDITIONAL OF p AND q

Valid & invalid argumentS
DEFINITION

Definition
An argument is a sequence of statements. All statements excluded the final one are called "hypotheses", the final statement is called "conclusion". A argument is the form:

p; q; r; ... $\quad \therefore f$ **(read therefore)**

An argument is valid means that if all hypotheses are true, the conclusion is also true.

Valid & invalid argumentS

EXAMPLE OF VALID FORM

EXAMPLE

Given an argument $p \vee (q \vee r)$ **;** $\neg r$ **;** $\therefore p \vee q$ **VALID**

p	q	r	$q \vee r$	$p \vee (q \vee r)$	$\neg r$	$p \vee q$
T	T	T	T	T	F	T
T	T	F	T	**T**	**T**	T
T	F	T	T	T	F	T
T	F	F	F	**T**	**T**	T
F	T	T	T	T	F	T
F	T	F	T	**T**	**T**	T
F	F	T	T	T	F	F
F	F	F	F	F	T	F

TRUE

SummarY

Theorem : Valid arguments,
given any propositions p,q **and** r**,**
the following arguments are valid:

MODUS PONENS

If it is raining, John does not go to school.
Now, it is raining.
CONCLUSION: John does not go to school.

$$p \to q \; ; p \; \therefore \; q$$

SummarY

Theorem : Valid arguments,
given any propositions p,q and r,
the following arguments are valid:

MODUS PONENS

MODUS TOLLENS

If it is raining, John does not go to school.
Now, John goes to school.
CONCLUSION: It is not raining.

$$p \to q \; ; \neg q \; \therefore \; \neg p$$

SummarY

Theorem : Valid arguments,
given any propositions p,q and r,
the following arguments are valid:

MODUS PONENS

MODUS TOLLENS

DISJUNCTION ADDITION

$$p \; \therefore \; p \vee q$$

SummarY

Theorem : Valid arguments,
given any propositions p,q and r,
the following arguments are valid:

MODUS PONENS

MODUS TOLLENS

DISJUNCTION ADDITION

CONJUNCTIVE SIMPLIFICATION

$$p \wedge q \;\therefore\; p$$

SummarY

Theorem : Valid arguments,
given any propositions p,q and r,
the following arguments are valid:

MODUS PONENS

MODUS TOLLENS

DISJUNCTION ADDITION

CONJUNCTIVE SIMPLIFICATION

CONJUNCTIVE ADDITION

$$p\,;\,q \;\therefore\; p \wedge q$$

SummarY

Theorem : Valid arguments,
given any propositions p,q and r,
the following arguments are valid:

MODUS PONENS **CONJUNCTIVE ADDITION**
MODUS TOLLENS **DISJUNCTIVE SYLLOGISM**
DISJUNCTION ADDITION
CONJUNCTIVE SIMPLIFICATION

Either Diana or John goes to school.
Diana does not go to school.
CONCLUSION: John goes to school.

$$p \vee q \; ; \neg q \therefore p$$

SummarY

Theorem : Valid arguments,
given any propositions p,q and r,
the following arguments are valid:

MODUS PONENS **CONJUNCTIVE ADDITION**
MODUS TOLLENS **DISJUNCTIVE SYLLOGISM**
DISJUNCTION ADDITION
CONJUNCTIVE SIMPLIFICATION
HYPOTHETICAL SYLLOGISM

$$p \to q \; ; q \to r \therefore p \to r$$

SummarY

Theorem : Valid arguments,
given any propositions p,q and r,
the following arguments are valid:

MODUS PONENS

CONJUNCTIVE ADDITION

MODUS TOLLENS

DISJUNCTIVE SYLLOGISM

DISJUNCTION ADDITION

CONJUNCTIVE SIMPLIFICATION

HYPOTHETICAL SYLLOGISM

DILEMMA

$$p \vee q \ ; p \rightarrow r \ ; q \rightarrow r \therefore \ r$$

ExampleS

1. Show that $(p\rightarrow q) \Leftrightarrow (\neg p\vee q)$ is a tautology.
2. Show that $(p\wedge q) \rightarrow (p\vee q)$ is a tautology.

ConsistenT

DEFINITION

A set of propositional expressions is consistent if there is an assignment of truth values to the variables in the expressions that makes each expression true.

EXAMPLE

1. $p \vee q$
2. $p \rightarrow r$
3. $q \rightarrow \neg r$
4. $(r \wedge s) \rightarrow q$

ConsistenT

EXERCISE

Show the truth values of each variable that this system is consistent if possible.

1. $r \rightarrow q$
2. $p \rightarrow q$
3. $\neg q \vee \neg r$
4. $\neg p \vee r$
5. $p \vee q$

ConsistenT

EXERCISE
Suppose that p,q,r is true and s is false.
IS this system consistent ?

1. $r \rightarrow q$
2. $p \rightarrow q$
3. $\neg q \vee \neg r$
4. $\neg p \vee s$
5. $p \vee q$

ExeRcIsEs

- Find the negations for these statements.
  - This computer program has a logical error in the first 10 lines or it is being run with an incomplete data set.
  - This exercise is easy but I cannot solve it in 5 minutes.
  - The dollar is at an all-time high and the stock market is at a record low.
- Which statement forms are tautologies by using truth tables.
  - $(p \wedge q) \vee (\neg p \vee (p \wedge \neg q))$
  - $(p \wedge \neg q) \wedge (\neg p \vee q)$
  - $((\neg p \wedge q) \wedge (q \wedge r)) \wedge \neg q$
  - $(\neg p \vee \neg q) \vee (p \wedge \neg q))$

EXERCISES

◆ Show that these statement forms are logically equivalences:

- $(p \rightarrow q) \wedge p$ $\qquad$ q
- $(p \vee q) \wedge \neg q$ $\qquad$ p
- $(p \vee q) \wedge ((\neg p \vee (q \wedge r)) \wedge (p \vee r)$ $\qquad$ $q \wedge r$
- $(p \wedge q) \vee r$ $\qquad$ $p \wedge (q \vee r)$

LOGIC

Consider the following statements:

All students go to school.
John is a student.
Diana is a student.
……

Of course we can conclude that

John goes to school.
Diana goes to school.
……

LoGIC

The statement "All students go to school" has two parts:
Variable students (denoted by variable x)
"go to school" (the predicate)

This statement can be denoted by P(x), where P denotes the predicate "go to school".
P(x) is said to be the value of the propositional function P at x.
Once a value has been assigned to the variable x, the statement P(x) becomes a proposition and has a truth value.

LoGIC

Predicate	: go to school.	P
Variable	: student	x
Constant	: John	a
	: Diana	b

P(a) is true.
P(b) is true.

PREDICATE LOGIC

- Predicate
- Quantifiers
- Negation
- Universal Conditional Statements
- Universal Modus Ponens
- Universal Modus Tollens

PREDICATE LOGIC

QUANTIFIERS

DEFINITION
The universal quantification of P(x) is the proposition
"P(x) is true for all values x in the universe of discourse".
It is denoted by $\forall x\ P(x)$.

DEFINITION
The existential quantification of P(x) is the proposition
"There exists an element x in the universe of discourse such that P(x) is true".
It is denoted by $\exists x\ P(x)$.

PREDICATE LOGIC

EXAMPLE

All integers are Real numbers.

Some integers are not odd.

These statements can be expressed as:

$\forall x(I(x) \rightarrow R(x))$

$\exists x(I(x) \wedge \neg O(x))$.

Where I(x) denotes "x is integer"

R(x) denotes "x is real", and

O(x) denotes "x is odd".

PREDICATE LOGIC

NEGATION

DEFINITION

The negation of a statement $\forall x P(x)$ is logically equivalent to a statement

$$\exists x \neg P(x).$$

DEFINITION

The negative of a statement $\exists x\ P(x)$ is logically equivalent to a statement

$$\forall x \neg P(x).$$

PREDICATE LOGIC

UNIVERSAL CONDITIONAL STATEMENTS

Consider a statement $\forall x\ P(x) \rightarrow Q(x)$.

CONVERSE
Its contrapositive is $\forall x\ \neg Q(x) \rightarrow \neg P(x)$.
INVERSE
Its inverse is $\forall x\ \neg P(x) \rightarrow \neg Q(x)$..
CONVERSE
Its converse is $\forall x\ Q(x) \rightarrow P(x)$.

PREDICATE LOGIC

UNIVERSAL MODUS PONENS

Consider a statement $\forall x\ P(x) \rightarrow Q(x)$.

For a particular e,

P(e) is true,

therefore Q(e) is true.

PREDICATE LOGIC

UNIVERSAL MODUS TOLLENS

Consider a statement $\forall x\ P(x) \rightarrow Q(x)$.

For a particular e,

$\neg Q(e)$ is true,

therefore $\neg P(e)$ is true.

PREDICATE LOGIC

RULES OF INFERENCES

Universal instantiation
$\forall x P(x)\ \therefore P(c)$ if $c \in U$.
Universal generalization
$P(c)$ for an arbitrary $c \in U \therefore \forall x P(x)$
Existential instantiation
$\exists x P(x) \therefore P(c)$ for some element $c \in U$
Existential generalization
$P(c)$ for some element $c \in U \therefore \exists x P(x)$

EXAMPLE

PREDICATE LOGIC

All students in this class are perfect.
Some students like “Automata”.
Some perfect people like “Automata”.
Let W be a set of people.
S(x) be “x is a student in this class”.
P(x) be “x is perfect”.
A(x) be “x like ‘Automata’ ”.

$\forall x \in W \quad (S(x) \rightarrow P(x))$

$\exists x \in W \quad (S(x) \wedge A(x))$

$\exists x \in W \quad (P(x) \wedge A(x))$

SET RELATION & FUNCTION

Contents

SOME DEFINITIONS

- **A is called a subset of B, denoted by $A \subseteq B$, if, and only if,**

  $\forall x$, if $x \in A$ then $x \in B$.
- **A is a proper subset of B, if, and only if,**

  $A \subseteq B$ and $A \neq B$.
- **Operations on sets**
  - **The union of A and B, $A \cup B$, is the set $\{ x \mid x \in A \text{ or } x \in B \}$.**
  - **The intersection of A and B, $A \cap B$, is the set $\{x \mid x \in A \text{ and } x \in B \}$.**
  - **The difference of B minus A, $B - A$, is the set $\{x \mid x \notin A \text{ and } x \in B \}$.**
  - **The complement of A, A^c, is the set $\{x \in \mathcal{U} \mid x \notin A \}$. [$\mathcal{U}$ = UNIVERSE].**

PropertieS
of sets

THEOREM

Given sets A,B and C.

- Commutative laws: $A \cap B = B \cap A$ $\quad$ $A \cup B = B \cup A$
- Associative laws: $(A \cap B) \cap C = A \cap (B \cap C)$
- Distributive laws: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- Idempotent laws: $A \cap U = A$ $\quad$ $A \cup U = U$
- De Morgan's laws: $(A \cup B)^c = A^c \cap B^c$ $\quad$ $(A \cap B)^c = A^c \cup B^c$
- Alternative representation for set difference $\quad$ $A-B = A \cap B^c$

SOME DEFINITIONS
CARTESIAN PRODUCT

DEFINITION

Given sets $A_1, A_2, A_3, \ldots, A_n$. The Cartesian products of $A_1, A_2, A_3, \ldots, A_n$, denote by $A_1 \times A_2 \times A_3 \times \ldots \times A_n$, is the set

$$\{(a_1, a_2, a_3, \ldots, a_n) \mid a_1 \in A_1.\ a_2 \in A_2.\ a_3 \in A_3. \ldots, a_n \in A_n\}.$$

EXAMPLE

Let A={ 1,2,3 }, B={ a,b } and C = { x,y },
the Cartesian products of A B and c is

{ (1,a,x), (1,a,y), (1,b,x), (1,b,y),
(2,a,x), (2,a,y), (2,b,x), (2,b,y),
(3,a,x), (3,a,y), (3,b,x), (3,b,y)}.

SOME DEFINITIONS
MUTUALLY/PAIRWISE DISJOINT

DEFINITION

Sets $A_1, A_2, A_3, \ldots, A_n$ are Mutually disjoint
(pairwise or nonoverlapping)

Iff, any two sets A_i, A_j with distinct subscripts have any elements in common, precisely $A_i \cap A_j$ = empty set $\emptyset$.

SOME DEFINITIONS
SET PARTITION

DEFINITION

A collection of nonempty sets $\{A_1, A_2, A_3, \ldots, A_n\}$ is a Partition of a set A *Iff*,

- $A = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$ and
- $A_1, A_2, A_3, \ldots, A_n$ are mutually disjoint.

SOME DEFINITIONS

POWER SET

DEFINITION

Give a set A, the power set of A, denoted by $\mathcal{P}$ (A), is the set of all subsets of A.

THEOREM

- For all sets A and B, if A $\subseteq$ B, then $\mathcal{P}$(A) $\subseteq$ $\mathcal{P}$(B).
- For all integers n $\geq$ 0, if a set A has n elements, then $\mathcal{P}$(A) has 2^n elements.

RELATIONS

BINARY

DEFINITION

Let A,B be sets. A binary relation $\mathcal{R}$ from A to B is a subset of the Cartesian product A$\times$B. Given (x,y), ordered pair, in A $\times$B, x is related to y by $\mathcal{R}$ written x$\mathcal{R}$y, *iff* (x,y)$\in$ $\mathcal{R}$.

EXAMPLE

The congruence modulo 2 relation

The relation $\mathcal{R}$ from Z to Z is defined as follows;

for all (x,y) $\in$ Z$\times$Z, x$\mathcal{R}$y *iff* x-y is even.

Example, 6$\mathcal{R}$2, 120$\mathcal{R}$36 etc.

RELATIONS

FUNCTION

DEFINITION

A function $\mathcal{F}$ from A to B is a relation from A to B, $\mathcal{F}$: A$\rightarrow$B, that satisfies the following properties:

1. For every x$\in$ A, there exists y$\in$ B such that (x,y)$\in$ $\mathcal{F}$.
2. For all x$\in$ A, and y, z$\in$ B,
   if (x,y)$\in$ $\mathcal{F}$ and (x,z)$\in$ $\mathcal{F}$ then y=z.

For (x,y) $\in$ $\mathcal{F}$, we usually write y = $\mathcal{F}$ (x) = image of x under $\mathcal{F}$,
and x is called pre-image of y under $\mathcal{F}$.

A is called domain of $\mathcal{F}$.

B is called co-domain of $\mathcal{F}$.

The set of all images of $\mathcal{F}$ is called range of $\mathcal{F}$.

RELATIONS

COMPOSITIONS OF FUNCTIONS

DEFINITION

A function f, g from A to B is a function from A to B.

$(f+g)(x) = f(x) + g(x).$
$(f\,g)(x) = f(x)\,g(x)$

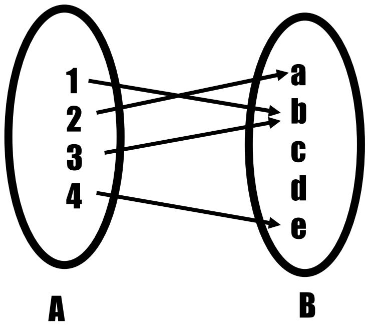
The composition of the functions f and g, denoted by $f \circ g$, is defined as

$(f \circ g)(x) = f(g(x))$

RELATIONS
FUNCTION

ARROW DIAGRAM

A function $\mathcal{F}$ from A to B.



$\mathcal{F}(1) = b$

$\mathcal{F}(2) = a$

$\mathcal{F}(3) = b$

$\mathcal{F}(4) = e$

FUNCTIONS
INJECTIVE

DEFINITION

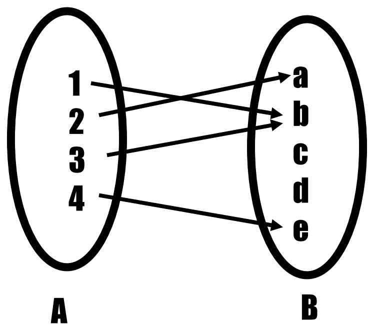
A function $\mathcal{F}$ from A to B is injective (or one-to-one)

iff for all elements x and y in A,

if $\mathcal{F}[x] = \mathcal{F}[y]$ then x = y.

Or, equivalently,

if $x \neq y$ then $\mathcal{F}[x] \neq \mathcal{F}[y]$.



This function is not One-to-one.

FUNCTIONS
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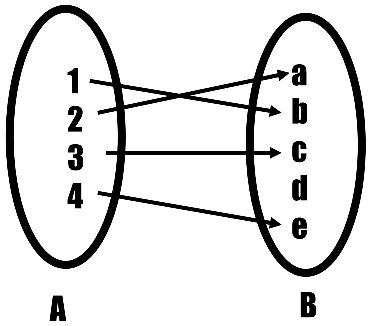


This function is One-to-one.

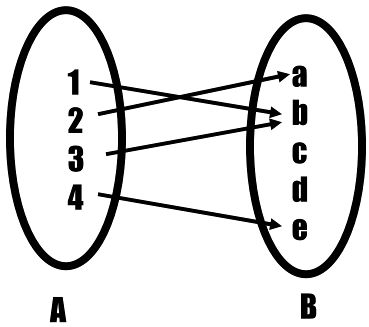
FUNCTIONS
SURJECTIVE

DEFINITION

A function $\mathcal{F}$ from A to B is surjective (or onto)

iff **for any element y in B, it is possible to find an element x in A such that**

$y = \mathcal{F}[x]$.



This function is not Onto: $c \in$ B but no element x in A that $\mathcal{F}[x]=c$.

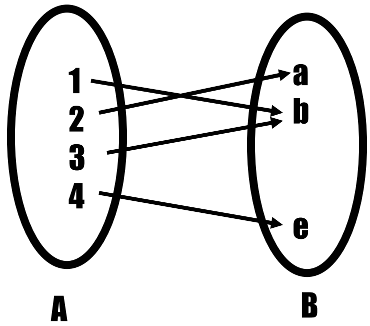
FUNCTIONS
SURJECTIVE

DEFINITION

A function $\mathcal{F}$ from A to B is surjective (or onto)

iff **for any element y in B, it is possible to find an element x in A such that**

y = $\mathcal{F}$(x) .

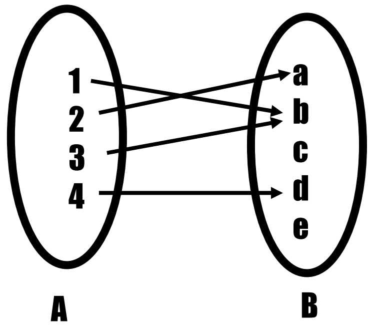


This function is Onto.

FUNCTIONS
BIJECTION

DEFINITION

A one-to-one correspondence (or bijection) $\mathcal{F}$ from A to B is a function that is both one-to-one and onto.



This function is not a bijection.

FUNCTIONS
BIJECTION
DEFINITION

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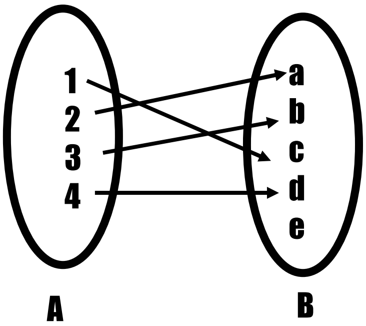

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FUNCTIONS
BIJECTION
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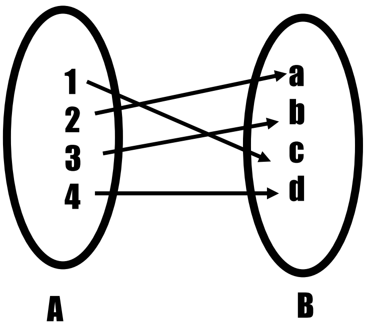

This function is bijection.

RELATIONS

THE INVERSE OF A RELATION

DEFINITION

Let $\mathcal{R}$ be a relation from A to B. Define the inverse relation, denoted by $\mathcal{R}^{-1}$ from B to A as follows:

$$\mathcal{R}^{-1} = \{ (y,x) \mid (x,y) \in \mathcal{R} \}.$$

RELATIONS

PROPERTIES

DEFINITION

Let $\mathcal{R}$ be a binary relation on A.

- $\mathcal{R}$ is reflexive *iff* for all $x \in A$, $x \mathcal{R} x$.
- $\mathcal{R}$ is symmetric *iff* for all $x,y \in A$, if $x \mathcal{R} y$ then $y \mathcal{R} x$.
- $\mathcal{R}$ is transitive *iff* for all $x,y,z \in A$,

  if $x \mathcal{R} y$ and $y \mathcal{R} z$ then $x \mathcal{R} z$.

EXAMPLE

The binary relation *"less than"* is transitive.

RELATIONS

EQUIVALENCE

DEFINITION

$\mathcal{R}$ is a equivalence relation on A *iff*

- $\mathcal{R}$ is a binary relation on A.
- $\mathcal{R}$ is reflexive .
- $\mathcal{R}$ is symmetric.
- $\mathcal{R}$ is transitive.

EXERCISE

Show that, the binary relation *"congreuence modulo 3"* is a equivalence relation.

RELATIONS

TRANSITIVE CLOSURE

DEFINITION

Let $\mathcal{R}$ be a binary relation on A.

The transitive closure of $\mathcal{R}$ is the binary relation $\mathcal{R}^t$ on A That satisfies the following three properties:

- $\mathcal{R}^t$ is transitive.
- $\mathcal{R} \subseteq \mathcal{R}^t$.
- $\mathcal{S}$ is any other transitive that contains $\mathcal{R}$ then $\mathcal{R}^t \subseteq \mathcal{S}$.

RELATIONS
PARTITION

DEFINITION

Given a partition of $A=\{A_1, A_2, A_3, \ldots, A_n\}$.
The binary relation induced by the partition, $\mathcal{R}$,
is defined on A as follows:
for all $x,y \in A$, $x \mathcal{R} y$ *Iff*, there is a subset A_j of the partition such that both x and y are in A_j.

THEOREM

Let A be a set with a partition and
Let $\mathcal{R}$ be the relation induced by the partition.
Then $\mathcal{R}$ is reflexive, symmetric and transitive.

RELATIONS
EQUIVALENCE CLASS

DEFINITION

Suppose A is a set and $\mathcal{R}$ is a equivalence relation on A.

For each $a \in A$, the equivalence class of a, denoted $[\ a\ ]$, is the set of all elements x in A such that
x is related to a by $\mathcal{R}$.

$$[\ a\ ] = \{\ x \in A \mid x\ \mathcal{R}\ a\ \}.$$

EXAMPLE

Let $\mathcal{R}$ be the relation *"congreuence modulo 3"*.

$[\ 3\ ] = \{\ 0, 3, -3, 6, -6, 9, -9, \ldots\ \}$.

RELATIONS

LEMMA 1

Let $\mathcal{R}$ be an equivalence relation on A, a b $\in$ A.

If a $\mathcal{R}$ b then [a] = [b].

THEOREM

LEMMA 2

Let $\mathcal{R}$ be an equivalence relation on A, a b $\in$ A, then

either $[a] \cap [b] = \emptyset$ or $[a] = [b]$.

THEOREM

If A is a nonempty set and $\mathcal{R}$ is an equivalence relation on A, then the distinct equivalence classes of $\mathcal{R}$ form a partition of A; that is, the union of the equivalence classes is all of A and the intersection of any two distinct classes is empty.

EXERCISES

- Let x and y be fractional numbers where

  x = a/b and y = c/d

  where a,b,c,d are integers.
- Let $\mathcal{R}$ be the relation defined as

  x $\mathcal{R}$ y *if, and only if*, $a \times d = b \times c$.

  Show that

  1. $\mathcal{R}$ is an equivalence relation and

  2. Describe the distinct equivalence classes of $\mathcal{R}$.