2110681 Computer Algorithms
Solution

1) $\sum_{k=1}^{n} \frac{1}{k} = O(\log n)$

That is $\sum_{k=1}^{n} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n}$

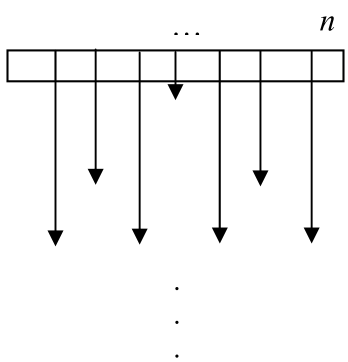
We rewrite; $= \left(\frac{1}{1+0}\right) + \left(\frac{1}{2+0} + \frac{1}{2+1}\right) + \left(\frac{1}{4+0} + \frac{1}{4+1} + \frac{1}{4+2} + \frac{1}{4+3}\right) + \ldots$

$$< \left(\frac{1}{2^0}\right) + \left(\frac{1}{2^1} + \frac{1}{2^1}\right) + \left(\frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{2^2}\right) + \ldots$$

$$= \sum_{i=0}^{\log_2 n} 1 \qquad = (\log_2 n) + 1$$

$$= O(\log n)$$

Thus $\sum_{k=1}^{n} \frac{1}{k} = O(\log n)$ #

2) 

Iteration		Prob		
1	$\times$	$1/n$		
2	$\times$	$2/n$		
3	$\times$	$2^2/n$	➔	$\sum$
. . .				
k	$\times$	$2^{k-1}/n$		

That is, $\sum_{k=1}^{\log_2 n} \frac{k2^{k-1}}{n}$

We know that, $\sum_{i=1}^{n} x^i = \frac{1-x^n}{1-x}$

Thus $$d/dx\left(\sum_{i=1}^{n} x^i\right) = d/dx\left(\frac{1-x^n}{1-x}\right)$$

$$\sum_{i=1}^{n} ix^{i-1} = \frac{(1-x)(-nx^{n-1}) + (1-x^n)}{(1-x)^2}$$

Replace $x = 2$, and $n = \log_2 n$

$$\sum_{k=1}^{\log_2 n} \frac{k2^{k-1}}{n} = \frac{(1-2)(-\log_2 n \cdot 2^{\log_2 n - 1}) + (1-2^{\log_2 n})}{n}$$

$$= \frac{\frac{n\log_2 n}{2} - n + 1}{n} \qquad = \Theta(\log n) \qquad \#$$

3) Homogeneous part: $a_n^{(h)} = 4a_{n-1} - 3a_{n-2}$

Characteristic equation χ: $r^2 = 4r - 3$

$r^2 - 4r + 3 = 0$

$0 = (r-1)(r-3)$ so, $r = 1, 3$

Thus, $a_n^{(h)} = \alpha_1 1^n + \alpha_2 3^n$

Particular part: $a_n^{(p)} = p_1 2^n + p_2 n^2 + p_3 n$

Then,

$p_1 2^n + p_2 n^2 + p_3 n = 4(p_1 2^{n-1} + p_2(n-1)^2 + p_3(n-1)) - 3(p_1 2^{n-2} + p_2(n-2)^2 + p_3(n-2)) + 2^n + n + 3$

$= \frac{4}{2} p_1 2^n + 4p_2 n^2 - 8p_2 n + 4p_2 + 4p_3 n - 4p_3 - \frac{3}{4} p_1 2^n - 3p_2 n^2 + 12p_2 n - 12p_2 - 3p_3 n + 6p_3 + 2^n + n + 3$

Then we have,

$p_1 2^n = \frac{4}{2} p_1 2^n - \frac{3}{4} p_1 2^n + 2^n$ ==> $p_1 = -4$

$p_2 n^2 = 4p_2 n^2 - 3p_2 n^2$ ==> $p_2 = p_2$

$p_3 n = -8p_2 n + 4p_3 n + 12p_2 n - 3p_3 n + n$ ==> $p_2 = -1/4$

and $0 = 4p_2 - 4p_3 - 12p_2 + 6p_3 + 3$ ==> $p_3 = -5/2$

Thus $a_n^{(p)} = -4 \cdot 2^n - \frac{1}{4} n^2 - \frac{5}{2} n$

Then we have

$$a_n = \alpha_1 1^n + \alpha_2 3^n - 4 \cdot 2^n - \frac{1}{4} n^2 - \frac{5}{2} n$$

Find α_1 and α_2

$a_0 = 1 = \alpha_1 + \alpha_2 - 4$

$5 = \alpha_1 + \alpha_2$, $\alpha_1 = 5 - \alpha_2$

$a_1 = 4 = \alpha_1 + 3\alpha_2 - 8 - 1/4 - 5/2$

$= 5 - \alpha_2 + 3\alpha_2 - 8 - 1/4 - 5/2$

$\alpha_2 = 39/8$

Thus $\alpha_1 = 1/8$

Hence, $a_n = \frac{1}{8} 1^n + \frac{39}{8} 3^n - 4 \cdot 2^n - \frac{1}{4} n^2 - \frac{5}{2} n$ #

4) Prove →

If f(x) = O(g(x)) then there exit constants c_1 and c_2

with $c_1|g(x)| \leq |f(x)| \leq c_2|g(x)|$

It follows that $|f(x)| \leq c_2|g(x)|$ and

$|g(x)| \leq 1/c_1|f(x)|$

Thus, $f(x) = O(g(x))$ and $g(x) = O(f(x))$ #

Prove ←

Suppose that $f(x) = O(g(x))$ and $g(x) = O(f(x))$

Then there are constants c_1 and c_2 such that

$|f(x)| \le c_1|g(x)|$ and

$|g(x)| \le c_2|f(x)|$ We can assume that $c_2 > 0$

$1/c_2|g(x)| \le |f(x)|$

Then we have

$1/c_2|g(x)| \le |f(x)| \le c_1|g(x)|$

Therefore, $f(x) = \Theta(g(x))$ #

5)

$$\lim_{n\to\infty}\frac{(\log n)^2}{n^{0.5}} = \lim_{n\to\infty}\frac{(\ln n/\ln 10)^2}{n^{0.5}} = \lim_{n\to\infty}\frac{(\frac{1}{\ln 10})^2(\ln n)^2}{n^{0.5}}$$

$$= (\frac{1}{\ln 10})^2 \lim_{n\to\infty}\frac{2(\ln n)(\frac{1}{n})}{0.5n^{-0.5}} = (\frac{1}{\ln 10})^2 \lim_{n\to\infty}\frac{2(\ln n)}{0.5n^{0.5}}$$

$$= (\frac{1}{\ln 10})^2 \lim_{n\to\infty}\frac{2\cdot\frac{1}{n}}{0.25n^{-0.5}} = 0$$

Thus $(\log n)^2 \prec n^{0.5}$ #