New Techniques for Computing Four-Finger Force-Closure Grasps of Polyhedral Objects

Attawith Sudsang and Jean Ponce
Beckman Institute
University of Illinois
Urbana, IL 61801, USA

Abstract: It was shown in [12] that four-finger force-closure grasps fall into three categories: concurrent, pencil, and regulus grasps. We propose new techniques for computing these three types of grasps. We have implemented them and present examples.

1 Introduction

We address the problem of computing stable four-finger grasps of three-dimensional polyhedral objects. More precisely, we consider force-closure grasps, such that arbitrary forces and torques exerted on the grasped object can be balanced by the contact forces exerted by the fingers, and assume hard-finger point contact with Coulomb friction.

There is an abundant literature on force-closure grasp planning: Kerr and Roth [4] give a numerical search algorithm for optimizing the equilibrium forces applied by the fingers, while Ji and Roth present an analytical method for computing these forces in the three-finger case [3]. Mishra, Schwartz, and Sharir [7] and Markenscoff, Ni, and Papadimitriou [5] propose linear-time algorithms for computing at least one force-closure grasp of a polyhedral object. Nguyen gives a geometric method for finding maximal independent two-finger grasps of polygons [8]. Omata uses fractional programming methods for manipulating polygonal objects with two fingers in the plane [9]. Ponce and Faverjon use linear programming to compute maximal independent three-finger grasps of polygonal objects [10]. Ferrari and Canny [2] and Mirtich and Canny [6] compute optimal grasps of polygonal objects by maximizing the set of external wrenches that can be balanced by the contact wrenches.

In [12], Ponce, Sullivan, Boissonnat and Merlet show that four-finger force-closure grasps fall into three categories: concurrent, pencil, and regulus grasps. In the case of concurrent grasps, they also give a sufficient condition for equilibrium which is linear in the position of the fingertips and of the point where the contact forces intersect. This allows them to compute all regions satisfying these constraints in the eight-dimensional configuration space of the grasp through projection and linear programming.

In this paper we give a new characterization of equilibrium which is valid for any four-finger grasp (Propositions 3 and 4). It is the basis for new methods for computing all three types of four-finger force-closure grasps. Like the approach proposed in [12], these techniques rely on linear programming, but by using a completely different grasp parameterization (in terms of *focus points* instead of fingertip positions), they avoid the projection step which is the most costly part of their predecessor.

We have implemented the proposed approach and present several examples (Figure 8). Mathematical details have been omitted for the sake of conciseness and can be found in [12, 13, 15].

2 Equilibrium and Force-Closure

We begin by recalling classical definitions and facts about force closure and equilibrium. We then give, without proof, two new propositions that characterize four-finger equilibrium (Propositions 3 and 4).

A hard finger in contact with some object at a point $\mathbf{x}$ exerts a force $\mathbf{f}$ with moment $\mathbf{x} \times \mathbf{f}$ with respect to the origin, but it cannot exert a pure torque. Force and moment are combined into a six-dimensional zero-pitch wrench $\mathbf{w} = (\mathbf{f}, \mathbf{x} \times \mathbf{f})$. Under Coulomb friction, the set of wrenches that can be applied by the finger is:

$$W = \{(\mathbf{f}, \mathbf{x} \times \mathbf{f}) : \mathbf{f} \in C\},$$

where C denotes the friction cone at $\mathbf{x}$.

A d-finger grasp is defined geometrically by the position $\mathbf{x}_i$ $(i = 1, .., d)$ of the fingers on the boundary of the grasped object. We can associate with each grasp the set of wrenches $\mathcal{W} \subset \mathbb{R}^6$ that can be exerted by the fingers. If we denote by W_i the wrench set associated with the i^{th} finger, we have

$$\mathcal{W} = \{\sum_{i=1}^{d} \mathbf{w}_i : \mathbf{w}_i \in W_i \text{ for } i = 1, \ldots, d\}.$$

Definition 1 *A grasp is said to achieve force closure when the corresponding wrench set $\mathcal{W}$ is equal to $\mathbb{R}^6$.*

In other words, a grasp achieves force closure when any external wrench can be balanced by wrenches at the fingertips. A somewhat weaker condition is equilibrium, defined below.

Definition 2 *A grasp is said to achieve equilibrium when there exist forces (not all of them being zero) in the friction cones at the fingertips such that the sum of the corresponding wrenches is zero.*

Formally, it is shown in [7, 8] for example that force closure implies equilibrium. More interestingly, the converse is also true for *non-marginal equilibrium* grasps, i.e., grasps such that the forces achieving equilibrium lie strictly inside the friction cones at the fingertips.

Proposition 1 [12] *A sufficient condition for four-finger force closure is non-marginal equilibrium.*

Thus, grasps achieving equilibrium with non-zero forces for some friction coefficient achieve force closure for any strictly greater friction coefficient.

The zero-pitch wrench $\mathbf{w} = (\mathbf{f}, \mathbf{x} \times \mathbf{f})$ associated with the force $\mathbf{f}$ can also be thought of as a coordinate vector for the line of action of this force. In this case, the six coordinates of $\mathbf{w}$ are called the Plücker coordinates of the line, and equilibrium trivially implies that the lines (or the Plücker vectors) associated with the contact forces are linearly dependent. Grassmann geometry [1] characterizes the varieties of various dimensions formed by sets of dependent lines, and it can be used to characterize four-finger equilibrium as follows (Figure 1).

Proposition 2 [1] *Four linearly-dependent lines either lie in a single plane, intersect in a single point, form two flat pencils having a line in common but lying in different planes, or form a regulus.*

The lines in a regulus lie on a doubly-ruled hyperboloid of one sheet (Figure 1(d)). A regulus can also be defined as the set of lines intersecting a fixed set of three skew lines (Figure 1(e)).

From now on we restrict our attention to non-planar grasps, i.e., to sets of contact forces whose lines

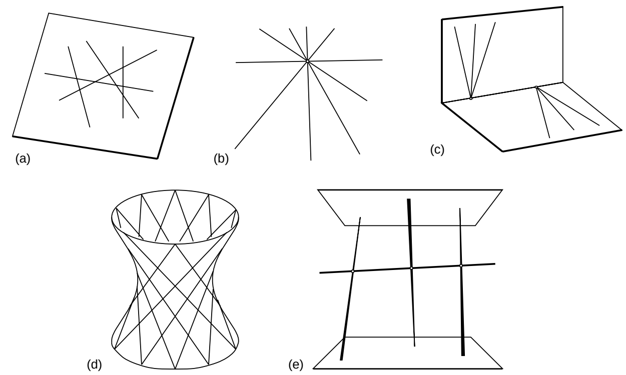

Figure 1. Configurations of linearly-dependent lines: (a) coplanar lines, (b) concurrent lines, (c) two flat pencils having a line in common, (d)-(e) a regulus.

of action do not all lie in the same plane. As noted in [12], Proposition 2 trivially implies that a necessary condition for non-planar equilibrium is that the contact forces all intersect in a point (*concurrent grasps*), lie in two flat pencils having a line in common (*pencil grasps*), or form a regulus (*regulus grasps*).

Figure 2 shows examples of each type of equilibrium grasp. In each case, the four forces have been grouped into pairs whose directions lie in two planes. In Figure 2(a), the four lines intersect in a point that lies in both planes, while in Figure 2(b) the two pairs of lines have been pulled apart, each pair of lines intersecting on the line formed by the intersection of the two planes. Finally, in the regulus case, the lines in each pair do not intersect anymore; they lie parallel to the original plane at a common distance from it (Figure 2(c)). It is intuitively obvious that the three grasps shown in Figure 2 achieve equilibrium, and this is easily confirmed by a simple calculation.

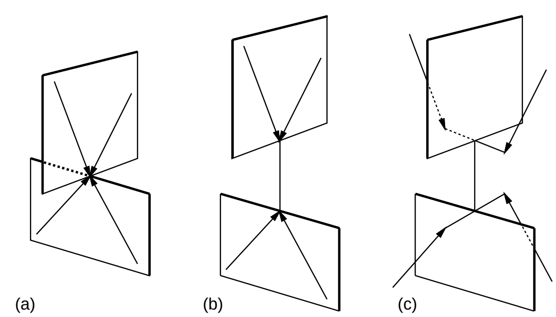

Figure 2. Examples of force configurations achieving equilibrium: (a) concurrent grasp, (b) pencil grasp, (c) regulus grasp. The contact faces are not shown.

Four-finger equilibrium grasps are completely characterized by the following proposition, which generalizes the condition given in [12].

Proposition 3 [13] *A necessary and sufficient condition for four non-coplanar points to form an equilibrium grasp with four non-zero contact forces is that: (P1) there exist four lines in the corresponding double-sided friction cones that either intersect in a single point, form two flat pencils having a line in common but lying in different planes, or form a regulus, and (P2) the vectors parallel to these lines and lying in the internal friction cones at the contact points positively span* $\mathbb{R}^3$.

We now give a sufficient (but not necessary) condition for equilibrium. We first need a definition.

Definition 3 *We say that four vectors θ-positively span $\mathbb{R}^3$ when, for any triple $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ of these vectors, the cones C_1, C_2, C_3 of half-angle θ centered on $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ lie in the interior of the same half-space and the cone $-C_4$ of half-angle θ centered on the direction opposite to the fourth vector $\mathbf{u}_4$ lies in the interior of the intersection of the trihedra formed by all triples of vectors belonging to C_1, C_2, and C_3 (Figure 3).*

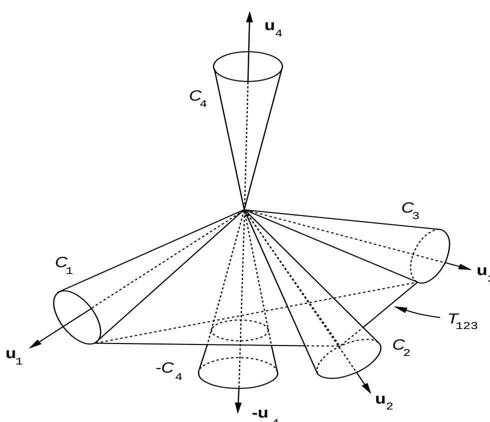

Figure 3. Four vectors θ-positively spanning $\mathbb{R}^3$. T_{123} is the intersection of the trihedra formed by all triples of vectors belonging to C_1, C_2, and C_3.

Clearly, any vector in $-C_4$ lies in the interior of the trihedron formed by any vectors in C_1, C_2, and C_3. If θ denotes the friction cone half-angle, the following proposition is a simple corollary of Proposition 3.

Proposition 4 [13] *A sufficient condition for four non-coplanar points to form an equilibrium grasp with four non-zero contact forces is that: (P1) there exist four lines in the corresponding double-sided friction cones that either intersect in a single point, form two flat pencils having a line in common but lying in different planes, or form a regulus, and (P3) the surface normals at the four contact points θ-positively span* $\mathbb{R}^3$.

The main advantage of Proposition 4 over Proposition 3 is that it replaces the condition (P2) – which depends on the actual contact forces' directions– by condition (P3) –which only depends on the normals to the grasped faces. This will allow us to use linear programming as a basis for the grasp planning approach presented in the next section: in each case, we will first select faces whose normals satisfy (P3), then compute the grasp configurations satisfying (P1).

3 Grasp Planning

We now present algorithms for computing concurrent, pencil, and regulus grasps. In each case, we parameterize the grasp by a set of *focus points*, from which we then compute the corresponding finger positions.

In both the concurrent and pencil cases, we actually compute *maximal independent regions* where the fingers can be positioned independently while maintaining force closure (a concept introduced by Nguyen in the two-dimensional two-finger case [8] and since then used in [10, 12, 13] for example). The results of our implementation are presented at the end of this section.

3.1 Concurrent Grasps

In this case the focus point is naturally the intersection $\mathbf{x}_0$ of the contact forces.

When the normals to the contact faces satisfy (P3), it follows immediately from Proposition 4 that a sufficient condition for equilibrium is that there exists a point in the intersection of the friction cones at the contact points. Equivalently, we obtain the following proposition.

Proposition 5 *A sufficient condition for four fingers to form an equilibrium grasp is that the four surface normals θ-positively span $\mathbb{R}^3$ and there exists a point $\mathbf{x}_0$ such that the inverted friction cones at this point intersect the four contact faces.*

A necessary and sufficient condition for the inverted cone at $\mathbf{x}_0$ to intersect a face F is that the projection $\mathbf{y}_0$ of $\mathbf{x}_0$ onto the plane P of the face F lies within the face grown by $d \tan\theta$, where d is the Euclidean distance between $\mathbf{x}_0$ and P (Figure 4).

As shown in Figure 4, the grown face is bounded by straight and circular edges. We can approximate each circular arc by a polygonal arc (this is equivalent to approximating a friction cone by a pyramid [12]), and, for convex faces, the condition is now linear in the coordinates of $\mathbf{x}_0$.

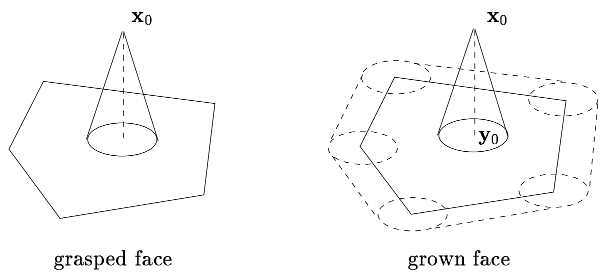

Figure 4. Writing that the inverted friction cone in $\mathbf{x}_0$ intersects the face is equivalent to writing that $\mathbf{x}_0$ projects inside the grown face

For a set of four faces whose normals θ-positively span $\mathbb{R}^3$, the set of equilibrium grasps is thus given by the set of points $\mathbf{x}_0$ satisfying the linear conditions associated with each face. Note that each valid point $\mathbf{x}_0$ yields four independent contact regions where fingers can be placed independently while ensuring force closure: these regions are simply the intersection of the inverted cones in $\mathbf{x}_0$ with the contact faces (Figure 5).

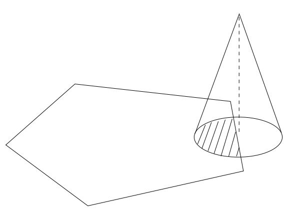

Figure 5. A contact region on face F.

We compute grasps which maximize the size of these regions. Unfortunately, the area of the contact region cannot be expressed linearly as a function of $\mathbf{x}_0$. A reasonable alternative is to use the largest disc contained in the region as a representative of the area. Finding the maximal contact regions reduces to finding a configuration maximizing the radius of the smallest of the four discs. This is indeed a linear program.

The main advantage of this method over the projection method presented in [12] is that all computations are performed in a three-dimensional configuration space instead of an eleven-dimensional one. Hence the new method is much more efficient. On the other hand, the maximal independent regions that it finds are usually smaller than the regions found by the projection algorithm.

3.2 Pencil Grasps

The idea of the construction is as follows: we construct two focus points $\mathbf{c}_1$ and $\mathbf{c}_2$ and project these points onto the contact faces. We then construct grasp discs centered at the projections such that for any quadruple of points in these discs we can construct a quadruple of forces achieving equilibrium (Figure 6).

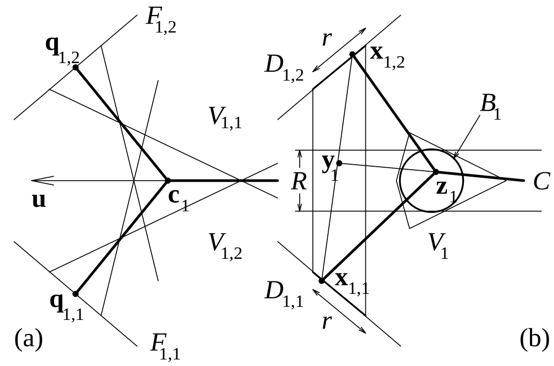

Figure 6. Notation for pencil grasps.

In the following, the subscript i is used to index objects associated with pencil number i ($i = 1, 2$), while the subscript j is used to index objects associated with face number j ($j = 1, 2$) of a given pencil.

We denote by P_i the vector plane perpendicular to both faces $F_{i,1}$ and $F_{i,2}$, and define the vector $\mathbf{u}$ as the intersection of P_1 and P_2. Let $\mathbf{x}_0$ denote some reference point, and k be some parameter, we define $\mathbf{c}_1 = \mathbf{x}_0 + k\mathbf{u}$, $\mathbf{c}_2 = \mathbf{x}_0 - k\mathbf{u}$. The points $\mathbf{c}_1$ and $\mathbf{c}_2$ are the two focus points of our grasp.

We denote by $\mathbf{q}_{i,j}$ the projection of the point $\mathbf{c}_i$ onto the face $F_{i,j}$ (Figure 6(a)). From Proposition 4, if the normals to the faces $F_{i,j}$ θ-positively span $\mathbb{R}^3$, there exist forces along the lines $(\mathbf{q}_{i,j}, \mathbf{c}_i)$ which achieve equilibrium.

Let $D_{i,j}$ denote the disc of radius r centered in $\mathbf{q}_{i,j}$. The four discs $D_{i,j}$ $(i, j = 1, 2)$ will be our four grasp discs (Figure 6(b)). Clearly, any line joining a point of $D_{i,1}$ to a point of $D_{i,2}$ lies in the cylinder C_i with radius r whose axis is the line joining $\mathbf{q}_{i,1}$ to $\mathbf{q}_{i,2}$.

We denote by $V_{i,j}$ the intersection of the pyramidal friction cones associated with points in $D_{i,j}$, and by V_i the intersection of $V_{i,1}$ and $V_{i,2}$. Clearly, $V_{i,1}$, $V_{i,2}$, and V_i are convex and non-empty (the point $\mathbf{c}_i$ belongs to all three convexes). If B_i denotes a ball of radius R centered in $\mathbf{c}_i$ and enclosed in V_i, any point in B_i lies in the friction cone of any point in the disc $D_{i,j}$.

Let us denote by C the cylinder of radius R tangent to the two balls B_1 and B_2. If $R \geq r$, and, for $i, j = 1, 2$, $\mathbf{x}_{i,j}$ denotes a point in $D_{i,j}$, we can construct equilibrium forces as follows. The line L_i joining $\mathbf{x}_{i,1}$ to $\mathbf{x}_{i,2}$ lies within C_i, and since both the axes of cylinders C_i and C lie in the plane P_i, L_i necessarily intersects C.

Let $\mathbf{y}_i$ denote a point in the intersection of L_i and C, and denote by L the line joining the points $\mathbf{y}_1$ and

$\mathbf{y}_2$. Clearly L intersects both B_1 and B_2, and if $\mathbf{z}_i$ is a point in the intersection of B_i and L, then the four lines $(\mathbf{x}_{1,1}, \mathbf{z}_1)$, $(\mathbf{x}_{1,2}, \mathbf{z}_1)$, $(\mathbf{x}_{2,1}, \mathbf{z}_2)$, $(\mathbf{x}_{2,2}, \mathbf{z}_2)$ form the desired pencils of lines.

It is easy to see that all of the constraints that must be satisfied by the variables involved in the construction of the pencils are linear, and maximal independent regions are found by maximizing r under these constraints.

3.3 Regulus Grasps

In this section we give a simple method for constructing regulus grasps that achieve equilibrium (Figure 7).

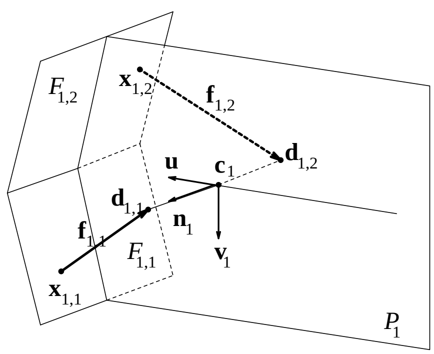

Figure 7. Notation for regulus grasps.

We use the following construction. Given four faces whose normals θ-positively span $\mathbb{R}^3$ and a reference point $\mathbf{x}_0$ we define the focus points $\mathbf{c}_1$ and $\mathbf{c}_2$ as before. We also define, for $i = 1, 2$, the points $\mathbf{d}_{i,1} = \mathbf{c}_i + l_i \mathbf{n}_i$ and $\mathbf{d}_{i,2} = \mathbf{c}_i - l_i \mathbf{n}_i$, where $\mathbf{n}_i$ is the unit normal to P_i.

Let $\mathbf{v}_i$ denote the unit vector such that $(\mathbf{u}, \mathbf{v}_i, \mathbf{n}_i)$ is a right-handed orthonormal basis, we define the directions

$$\begin{cases} \mathbf{f}_{i,1} = x_i \mathbf{u} + y_i \mathbf{v}_i + z_i \mathbf{n}_i, \\ \mathbf{f}_{i,2} = x_i \mathbf{u} - y_i \mathbf{v}_i - z_i \mathbf{n}_i, \end{cases}$$

and the corresponding contact points

$$\begin{cases} \mathbf{x}_{i,1} = \mathbf{d}_{i,1} - \mathbf{f}_{i,1}, \\ \mathbf{x}_{i,2} = \mathbf{d}_{i,2} - \mathbf{f}_{i,2}. \end{cases}$$

Clearly, the four lines passing through the points $\mathbf{x}_{i,j}$ with directions $\mathbf{f}_{i,j}$ are linearly dependent. We can therefore construct equilibrium regulus grasps through linear programming: we minimize the maximum of the L^∞ distances between the contact points and the centers of the contact faces under the constraints that the points $\mathbf{x}_{i,j}$ belong to the contact faces and the directions $\mathbf{f}_{i,j}$ belong to the corresponding pyramidal friction cones. These constraints are indeed linear in all the parameters.

It should be noted that this method allows us to construct grasps that would not achieve equilibrium in the frictionless case (the directions $\mathbf{f}_{i,j}$ are not necessarily parallel to the faces' normals), but that, so far, we have not been able to come up with a method for constructing independent grasp regions in the regulus case.

3.4 Implementation and Results

We have implemented the three algorithms in C, using the simplex routine from *Numerical Recipes in C* [14] for linear programming. Figure 8 shows some examples of results. In all cases, we have used a value of ten degrees for the angle θ, and the fingers are shown as long dark lines. In the concurrent and pencil cases, the fingers have been positioned at the center of the corresponding independent grasp regions. Our programs took less than ten seconds of CPU time on a SUN SPARCstation 10 to compute all of the force-closure grasps of each object.

4 Discussion

We have presented a geometric characterization of four-finger equilibrium and force closure, given algorithms for computing concurrent, pencil, and regulus grasps of polyhedral objects, and demonstrated efficient implementations of these algorithms. Let us conclude by discussing some of the issues raised by our work and by sketching some future research directions.

We believe that pencil and regulus grasps will prove useful in practice: for example pencil grasps may be used to manipulate elongated objects for which concurrent grasps do not exist, maybe using two cooperating robots equipped with simple two-finger grippers. One can also imagine manipulation tasks where two pairs of fingers twist a screw and the mating nut in opposite directions until a regulus grasp is obtained.

It is clear that our approach to pencil and regulus grasp computation is limited to a subset of these grasps: in both cases, we have chosen an arbitrary direction for the line joining the focus points. This choice was motivated by our use of linear programming as a basis for grasp planning. Likewise, we have not been able to compute maximal independent regions in the regulus case using linear programming. An alternative would be to use non-linear cell decomposition techniques to characterize all regions of the grasp configuration space that yield stable pencil and regulus grasps (see [11] for a similar approach in the two-dimensional, two-finger case).

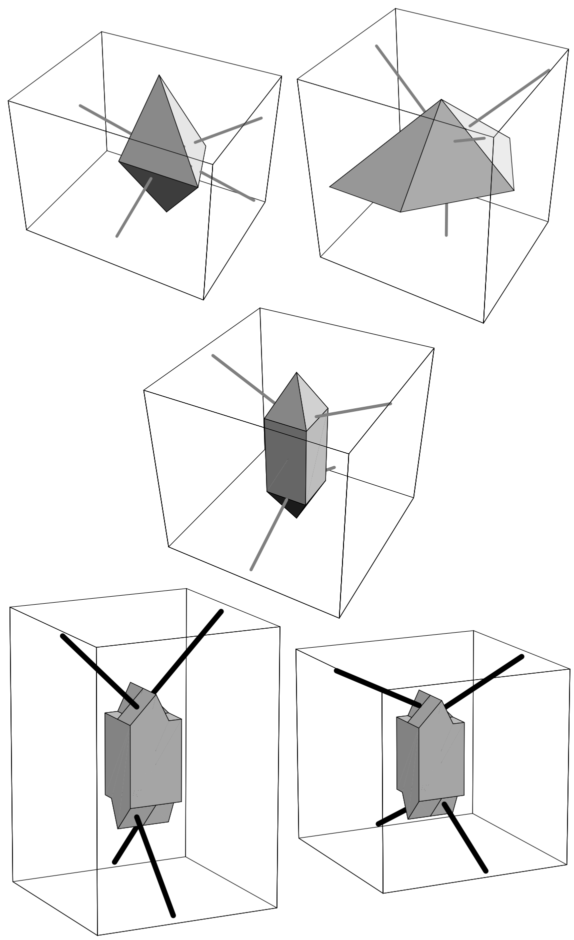

Figure 8. Examples of grasps computed by our implementation: two concurrent grasps (top), a pencil grasp (middle), two regulus grasps (bottom).

It would also be interesting to compute four-finger grasps that are optimal according to some functional consideration: for example, given a fixed set of contact points and some bound on the magnitude of the contact forces, Ferarri and Canny [2] propose to compute a maximal ball centered at the origin and contained in the convex formed by the contact wrenches; clearly, the contact forces can generate any wrench contained in this ball, and its radius provides a measure of the grasp's efficiency. Computing a four-finger grasp configuration which is optimal according to this criterion is a much more challenging non-linear problem that we plan to address in the future.

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