Nested Quantifiers

- Readings:

  Rosen Section 1.4

Nested Quantifiers

- Quantifiers that occur within the scope of other quantifiers.
- E.g.:

  $\forall x \forall y((x > 0) \wedge (y < 0) \rightarrow (xy < 0))$

Nested Quantifiers

Statement	… is TRUE when
$\forall x \forall y\ P(x,y)$ $\forall y \forall x\ P(x,y)$	P(x,y) is true for every pair of x,y
$\forall x \exists y\ P(x,y)$	For every x, there is a y for which P(x,y) is true.
$\exists x \forall y\ P(x,y)$	There is an x for which P(x,y) is true for every y.
$\exists x \exists y\ P(x,y)$ $\exists y \exists x\ P(x,y)$	There is a pair x,y for which P(x,y) is true.

Negating Nested Quantifiers

- Successively applying the rules for negating statements involving a single quntifier.
- Example (Rosen p.48):

$$\neg \forall x \exists y\ (xy = 1)\ \equiv \exists x\ \neg \exists y\ (xy = 1)$$
$$\equiv \exists x\ \forall y\ \neg(xy = 1)$$
$$\equiv \exists x\ \forall y\ (xy \neq 1)$$

Relations

- Rosen: Section 7.1

Relations

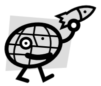
- A (binary) relation form A to B is a subset of AxB
- A *relation on the set A* is a *relation from A to A*
- A function from A to B is a relation from A to B
- <u>Examples</u>:
  $R_1 = \{(1,1),(1,2),(2,1),(2,3)\}$
  $R_2 = \{(a,b) \mid a = b \text{ or } a = -b\}$
  a and b are integers

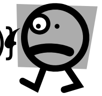
Properties of Relations

- R on the set A is *reflexive* $\leftrightarrow \forall a\ ((a,a) \in R)$

<u>Example</u>: Consider relations on {1,2,3,4}

R must contain (1,1),(2,2),(3,3),(4,4)

R1 = {(1,1),(1,2),(1,3),(2,2),(3,3),(4,1),(4,4)} 

R2 = {(1,1),(2,1),(2,3),(3,1),(3,2),(3,3),(3,4),(4,4)} 

Symmetric and Antisymmetric

- R on a set A is *symmetric*
  $\leftrightarrow \forall a \forall b(\ (a,b) \in R \rightarrow (b,a) \in R)$
- R on a set A is *antisymmetric*
  $\leftrightarrow \forall a \forall b(\ ((a,b) \in R \wedge (b,a) \in R) \rightarrow (a=b)\ )$
- These two are NOT opposite.

Symmetric and Antisymmetric

- *Symmetric* ↔ $\forall a \forall b(\ (a,b) \in R \rightarrow (b,a) \in R)$
- *Antisym.* ↔ $\forall a \forall b(\ ((a,b) \in R \wedge (b,a) \in R) \rightarrow (a=b)\ )$

Example:

	Sym	Antisym
$R_1 = \{(1,1),(1,2),(2,1)\}$	☐	☐
$R_2 = \{(1,1),(1,2)\}$	☐	☐
$R_3 = \{(a,b) \mid a = b\}$ (on Int.)	☐	☐
$R_4 = \{(2,1)\}$	☐	☐
$R_5 = \{(a,b) \mid a + b \leq 3\}$ (on Int.)	☐	☐

Transitive Relations

- R on a set A is *transitive* $\leftrightarrow \forall a \forall b \forall c(\ ((a,b) \in R \wedge (b,c) \in R) \rightarrow (a,c) \in R)$

Example:

$R_1 = \{(1,2),(2,3),(1,3),(1,4)\}$
$R_2 = \{(1,1),(1,2),(1,3),(2,4)\}$
$R_3 = \{(a,b) \mid a < b\}$

Combining Relations

- Since a relation is a set, we can apply all set operators to relations.
- Example (Rosen p.477)

  $R_1 = \{(1,1),(2,2),(3,3)\}$,
  $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$

  $R_1 \cap R_2 = \{(1,1)\}$
  $R_1 - R_2 = \{(2,2),(3,3)\}$

Composite Relations

- R is a relation from A to B
- S is a relation from B to C
- $S \circ R = \{(a,c) \mid a \in A, c \in C$, and there exists $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S\}$

Composite Relations

- Example (Rosen p.478):

  R is a relation from {1,2,3} to {1,2,3,4} with R={(1,1),(1,4),(2,3),(3,1),(3,4)} and S is a relation from {1,2,3,4} to {0,1,2} with S={(1,0),(2,0),(3,1),(3,2),(4,1)}.

  What is the composite of R and S?

SoR = {(1,0),(1,1),(2,1),(2,2),(3,0),(3,1)}

Composite Relations

$R^{n+1} = R^n \circ R$ and $R^1 = R$

R^n is transitive $\leftrightarrow R^n \subseteq R$