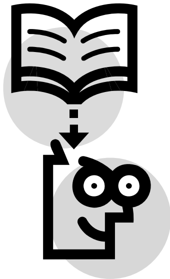
Methods of Proof

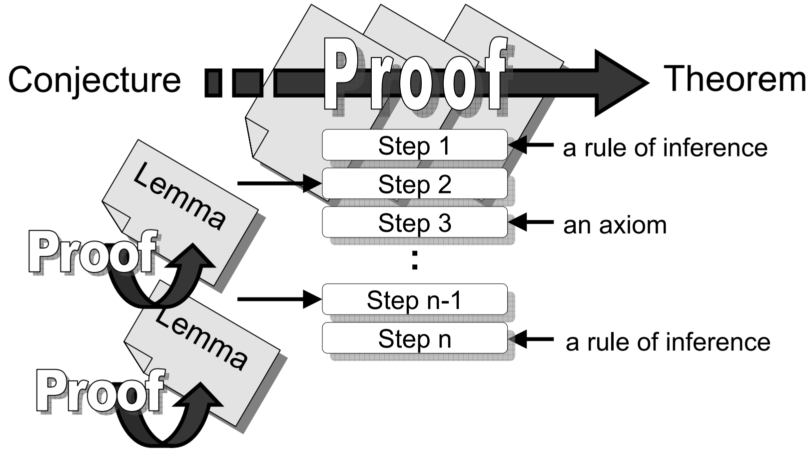
- Readings:



Rosen Section 1.5

Atiwong Suchato
Faculty of Engineering, Chulalongkorn University

Proof Mechanisms



Atiwong Suchato
Faculty of Engineering, Chulalongkorn University

Rules of Inference

- Provide justification of the steps used to show that *a conclusion follows a set of hypotheses*.
- Each uses *a tautology* as its basis.
- E.g.:

  The *law of detachment* or *Modus ponens*

  p

  $\underline{p \rightarrow q}$

  $\therefore q$

  (Based on $(p \wedge (p \rightarrow q)) \rightarrow q$)

Atiwong Suchato
Faculty of Engineering, Chulalongkorn University

Rules of Inference

Addition	$\underline{p\quad .}$ $\therefore p \vee q$
Simplification	$\underline{p \wedge q}$ $\therefore p$
Conjunction	p $\underline{q\quad .}$ $\therefore p \wedge q$
Modus ponen	p $\underline{q \rightarrow q}$ $\therefore q$

Modus tollens	$\neg q$ $\underline{p \rightarrow q}$ $\therefore \neg p$
Hypothetical syllogism	$p \rightarrow q$ $\underline{q \rightarrow r}$ $\therefore q \rightarrow r$
Disjunction syllogism	$p \vee q$ $\underline{\neg p\quad .}$ $\therefore q$
Resolution	$p \vee q$ $\underline{\neg p \vee r}$ $\therefore q \vee r$

Atiwong Suchato
Faculty of Engineering, Chulalongkorn University

Rules of Inference

- Example (Rosen p.57):

  If it rains today, we will not have a barbecue today.

  If we do not have a barbecue today, we will have it tomorrow

  Therefore, if it rains today, then we will have a barbecue tomorrow.

  Which rule of inference is used?

Rules of Inference

- Example:

  If it floods today, Chula will close.

  Chula is not closed today.

  Therefore, it did not flood today.

  Which rule of inference is used?

Valid Arguments

- An argument is called ***valid*** if whenever all the hypotheses are true, the conclusion is also true.

  Showing that $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q$ is true.

Valid Arguments

- Example (Rosen p.59):

h_1: If you send me an email, I will finish writing this program.

h_2: If you do not send me an email, I will go to bed early.

h_3: If I go to bed early, I will wake up feeling refreshed.

Lead to?: *If I do not finish writing program, then I will wake up feeling refreshed.*

Valid Arguments

- Example (Rosen p.60):
  Show that **(p ∧ q)** ∨ **r** and **r→s** imply **p** ∨ **s**

Rules of Inference: Quantified Statements

Universal Instantiation	∀xP(x) ∴ P(c)
Universal Generalization	P(c) for an arbitrary c ∴ ∀xP(x)
Existential Instantiation	∃xP(x) ∴ P(c) for some element c
Existential Generalization	P(c) for some element c ∴ ∃xP(x)

Rules of Inference: Quantified Statements

- Example (Rosen p.62):

Show that:

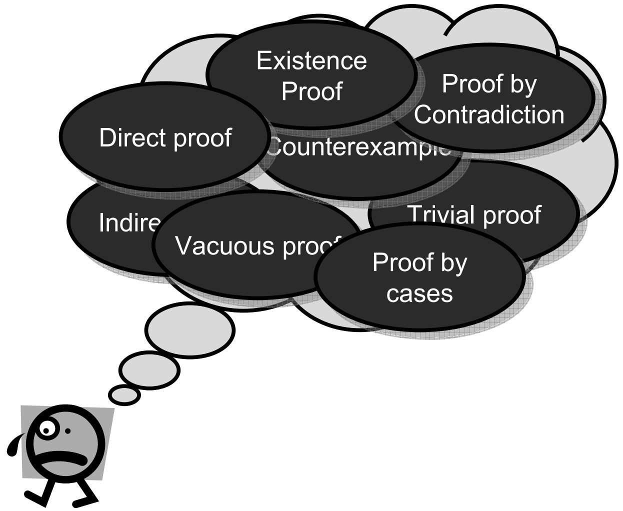
A student in this class has not read the book.

Everyone in this class passed the first exam.

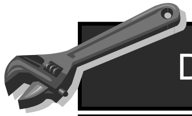
imply:

Someone who passed the first exam has not read the book.

Methods of Proving Theorems



Proving p→q



Direct Proof

Indirect Proof

Vacuous Proof

Trivial Proof

Proof by Contradiction

Atiwong Suchato
Faculty of Engineering, Chulalongkorn University

Proving p→q

Direct Proof — Show that if **p** is true, **q** must be true.

Indirect Proof — $p \rightarrow q \equiv \neg q \rightarrow \neg p$

Show that if **¬q** is true, **¬p** must be true.

Vacuous Proof

Trivial Proof

Proof by Contradiction

Atiwong Suchato
Faculty of Engineering, Chulalongkorn University

Proving p→q

- <u>Example</u> (Rosen p.64):
  Show that "If *n* is an odd integer, n^2 is an odd integer"

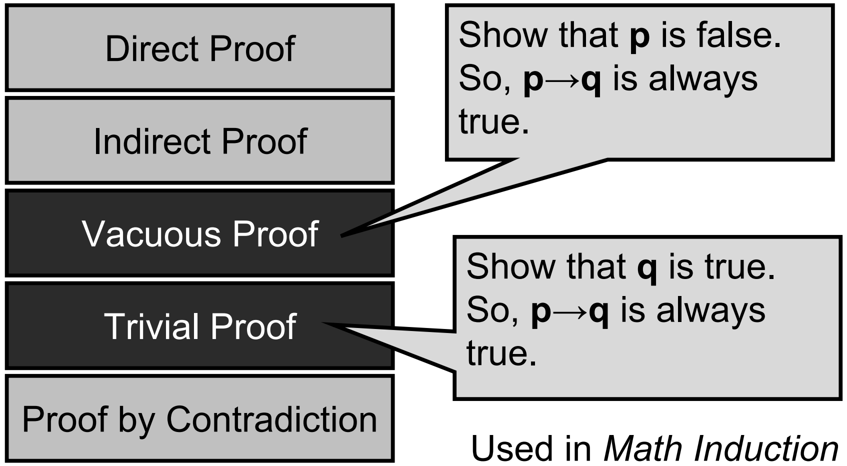
Atiwong Suchato
Faculty of Engineering, Chulalongkorn University

Proving p→q

- <u>Example</u> (Rosen p.64):
  Show that "If *n* is an integer and n^2 is odd, then *n* is odd."

Atiwong Suchato
Faculty of Engineering, Chulalongkorn University

Proving p→q



Proving p→q

- Example (Rosen p.64):
  P(n) = "If $n > 1$,then $n^2 > n$"
  Show that P(0) is true.

- Example (Rosen p.64):
  P(n) = "If a and b are positive integers with $a \geq b$, then $a^n \geq b^n$"
  Show that P(0) is true.

Proof by Contradiction

- Proof by Contradiction
  - Suppose we want to prove a statement ***s***
  - Start by assuming $\neg s$ is true.
  - Show that $\neg s$ implies a contradiction. ($\neg s \rightarrow F$)
  - Then, $\neg s$ must be false (or ***s*** must be true).

Proof by Contradiction

- Example:
  Show that at least 10 of any 64 days chosen must fall on the same day of the week.

Proof p→q by Contradiction

- <u>Proof by Contradiction</u>
  - Start by assuming $\neg$ ***(p→q)*** is true.
  - That means $\boldsymbol{p \wedge \neg q}$ is true.
    (since $\neg (p \rightarrow q) \equiv \neg(\neg p \vee q) \equiv p \wedge \neg q$)
  - Show that $\boldsymbol{p \wedge \neg q}$ is a contradiction
  - Then, $\neg$ ***(p→q)*** must be false
    (or ***(p→q)*** must be true).

Atiwong Suchato
Faculty of Engineering, Chulalongkorn University

Proving p→q

- <u>Example</u>:
  Prove that "If *n* is an integer and n^3+5 is odd, then *n* is even". Using:
  (a) an indirect proof.
  (b) a proof by contradiction.

Atiwong Suchato
Faculty of Engineering, Chulalongkorn University

Proof by Cases

- Prove an implication of the form:

$$(p_1 \vee p_2 \vee \ldots \vee p_n) \rightarrow q$$

by proving that:

$$p_i \rightarrow q,\ i = 1,2, \ldots, n$$

Atiwong Suchato
Faculty of Engineering, Chulalongkorn University

Proof by Cases

- <u>Example</u> (Rosen p.67):
  Show that $|xy| = |x||y|$, where *x* and *y* are real numbers.

Atiwong Suchato
Faculty of Engineering, Chulalongkorn University

Proof of $p \leftrightarrow q$

- Since $(p \leftrightarrow q) \leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$, then *prove both* $p \rightarrow q$ *and* $q \rightarrow p$
- Equivalent propositions ($p_1 \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_n$) are proven by *proving* $p_1 \rightarrow p_2, p_2 \rightarrow p_3, \ldots, p_n \rightarrow p_1$

Equivalent Propositions

- Example (Rosen p.68):

  Show that these statements are equivalent:

  p_1: *n* is an even integer.

  p_2: *n -1* is an odd integer.

  p_3: n^2 is an even integer.

Proof of Proposition Involving Quantifiers

- Existence proofs: A proof of $\exists x P(x)$
- *Constructive existence proof:*
  - Find an element *c* such that P(*c*) is true.
- *Non-constructive existence proof:*
  - Do not find an element *c* such that P(*c*) is true, but use some other ways.

Existence Proofs

- Example (Rosen p.69):

  Show that $\exists x \exists y$ (x^y is rational.)

Proof of Proposition Involving Quantifiers

- Uniqueness proofs: showing that there is a unique element x such that P(x).

  1) *Existence*:

     Show that $\exists x P(x)$

  2) *Uniqueness*:

     Show that if $y \neq x$, P(y) is false.
- is the same as proving:

$$\exists x(P(x) \wedge \forall y(y \neq x \rightarrow \neg P(y)))$$

Uniqueness Proofs

- Example (Rosen p.70):

  Show every integer has a unique additive inverse. (If p is an integer, there exists a unique integer q such that p+q = 0.)

Atiwong Suchato
Faculty of Engineering, Chulalongkorn University

Counterexamples

- Show that $\forall x P(x)$ is false.

- Example (Rosen p.70):

  "Every positive integer is the sum of the squares of three integers" ??

Atiwong Suchato
Faculty of Engineering, Chulalongkorn University