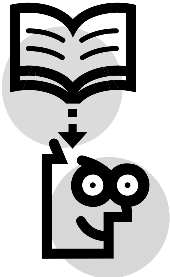
Proof Strategy & Mathematical Induction



- Readings:
  Proof Strategy:
  Rosen Section 3.1
  Mathematical Induction:
  Rosen Section 3.3

Forward & Backward Reasoning

- Forward reasoning:
  - To prove $p \rightarrow q$ (or $\neg q \rightarrow \neg p$):
    - Start with p (or $\neg q$)
    - Use axioms + known theorems + etc. in steps.
    - Lands the conclusion q (or $\neg p$).
  - Works with simple results.
- Backward reasoning:
  - Start with the conclusion instead.

Forward & Backward Reasoning

- Example (Rosen p.215):
  If a and b are distinct positive real numbers,
  $(a+b) / 2 > \sqrt{ab}$

Proof by Cases

- Example (Rosen p.216):
  If n is an integer not divisible by *2* or *3*, then n^2-1 is divisible by *24*.

- Example:
  Show that there are no integers x and y such that $3x^2 - 8y = 1$

- Example:
  Prove that there are infinitely many primes.

- Example (Rosen p.217):
  Prove that there are infinitely many primes of the form $4k+3$, where k is a nonnegative integer.

Mathematical Induction

- A proof by induction that $P(n)$ is true for every positive integer n consists of 2 steps:

BASIC STEP: Show that $P(1)$ is true.

INDUCTIVE STEP:
Show that $P(k) \rightarrow P(k+1)$ is true for every positive integer k

- Example (Rosen p.240):
  Prove that the sum of the first *n* odd positive integers is n^2.

  P(n):

  Basic Step:

  Inductive Step:

- Example (Rosen p.241):
  Prove that $n < 2^n$ for all positive integers *n*.

  P(n):

  Basic Step:

  Inductive Step:

- Example (Rosen p.241):
  Prove that n^3-*n* is divisible by *3* all positive integers *n*.

  P(n):

  Basic Step:

  Inductive Step:

Mathematical Induction

- Sometimes we want to prove that *P(n)* is true for *n = b, b+1, b+2, …* where *b* is an integer other than 1.

BASIC STEP: Show that *P(b)* is true.

INDUCTIVE STEP:
Show that $P(k) \rightarrow P(k+1)$ is true for every positive integer *k*

Example (Rosen p.243):

Prove that $H_{2^n} \geq 1 + \frac{n}{2}$ $\quad H_j = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}$

whenever n is a nonnegative integer.

P(n):

Basic Step:

Inductive Step:

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Proving Mathematical Induction

- The well-ordering property:

> Every nonempty set of nonnegative integers has a least element.

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Proving Mathematical Induction

- Show that *P(n)* must be true for all positive integers when *P(1)* and *P(k)→P(k+1)* are true.
- Assume that *P(n)* is not true for at least a positive integer. Then, the set *S* for which *P(n)* is false is nonempty.
- *S* has the least element, called *m*. ($m \neq 1$)
- Since $m-1 < m$, then $m-1 \notin S$ (or *P(m-1)* is true)
- But *P(m-1)→P(m)* is true. So, *P(m)* must be true.
- This contradicts the choice of *m*.

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Strong Induction

- A proof by induction that *P(n)* is true for every positive integer *n* consists of 2 steps:
- Use a different induction step.

BASIC STEP: Show that *P(1)* is true.

INDUCTIVE STEP:
Show that $[P(1) \wedge P(2) \wedge \ldots \wedge P(k)] \rightarrow P(k+1)$ is true for every positive integer *k*

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- Example (Rosen p.250):

  Show that if *n* is an integer greater than *1*, then *n* can be written as the product of primes.

  P(n):

  Basic Step:

  Inductive Step:

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- Example (Rosen p.250):

  Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

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