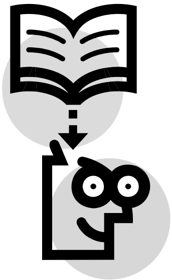
Counting II



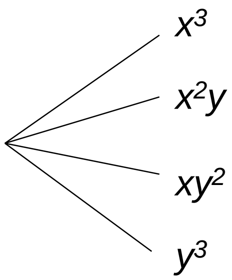
- Readings:
  Basic counting principles
  - Rosen section 4.4-4.6

The Binomial Theorem

- A binomial expression is the sum of two terms, such as $(x + y)$
- The binomial theorem concerns the expansion of powers of binomial expressions.
- E.g.: $(x+y)^3 = x^3+3x^2y+3xy^2+y^3$

The Binomial Theorem

$(x+y)^3 = (x+y)(x+y)(x+y)$



- C(n,r) is also called a ***binomial coefficient***.

The Binomial Theorem

$$(x+y)^n = \sum_{j=0}^{n} c(n,j)x^{n-j}y^j$$
$$= c(n,0)x^n + c(n,1)x^{n-1}y + c(n,2)x^{n-2}y^2 +$$
$$\ldots + c(n,n-1)xy^{n-1} + c(n,n)y^n$$

Example (Rosen p.328):
$(x+y)^4=$

What is the coef. of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

- Example (Rosen p.328):
  What is the coef. of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

- Show that:
  Let n be a positive integer. Then:

$$\sum_{k=0}^{n} (-1)^k c(n,k) = 0$$

- Show that:
  Let n be a nonnegative integer. Then:

$$\sum_{k=0}^{n} 2^k c(n,k) = 3^n$$

Pascal's Identity

Let n and k be positive integers with $n \geq k$, Then
$c(n+1,k) = c(n,k-1)+c(n,k)$

Proof:

Pascal's Triangle

- ***Pascal's identity*** together with ***initial conditions*** *c(n,0) =c(n,n)=1* for all integers *n* can be used to *recursively define binomial coefficients*.

Vandermonde's Identity

- Let *m*, *n*, and *r* be nonnegative integers with *r* not exceeding either *m* or *n*. Then:

$$c(m+n,r)=\sum_{k=0}^{r}c(m,r-k)c(n,k)$$

Proof:

- Show that:

  If *n* is a nonnegative integer. Then:

$$c(2n,n)=\sum_{k=0}^{n}c(n,k)^2$$

- Show that:

  Let *n* and *r* be nonnegative integers with $r \leq n$. Then:

$$c(n+1,r+1)=\sum_{j=r}^{n}c(j,r)$$

Generalized Permutations and Combinations

- Permutations with repetition
- Combinations with repetition
- Permutations with indistinguishable objects
- Distributing objects into boxes

Permutations with Repetition

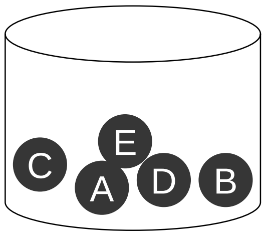
- Easily done using product rule.
- Example:

  How many strings of length *n* can be formed from the English alphabets, if each alphabet can be used no more than once?

  How many strings can be formed, if repetition is allowed?

- Example:

  A bucket containing 5 different balls. We pick a series of 3 balls randomly from the bucket.

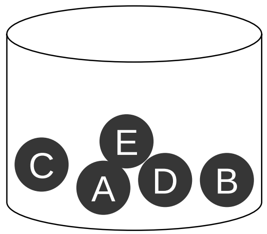


How many sequences of 3 balls are there that we can draw from the bucket? (without putting any balls back in)

How many sequences if we put the drawn ball back in before we draw another ball?

- Example:

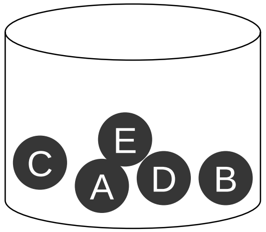
  A bucket containing 5 different balls. We pick 3 balls randomly from the bucket at the same time.



How many ways can the balls be selected?

- Example:

  A bucket containing 5 different balls. We pick 3 balls randomly from the bucket at the same time.



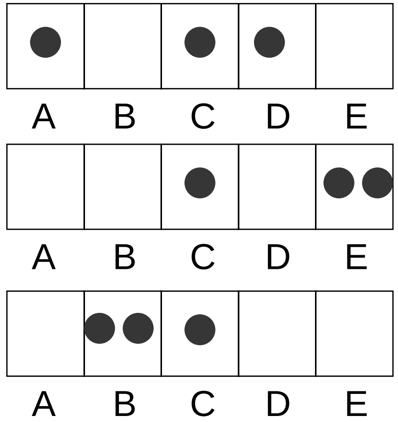
How many ways can the balls be selected, if repetitions are allowed?

i.e.: We draw one ball at a time and put the drawn ball back in before drawing another one, while we do not care about the order.

or

Instead of only five balls in the bucket, there are five types of balls where there are more than 3 balls for each type.

Combinations with Repetition



Combinations with Repetition

- There are ***c(n-1+r,r)*** *r-combinations* from a set with n elements when repetition of elements is allowed.
- Example (Rosen p.338):

  There are 4 types of cookies in a cookie shop. How many ways can 6 cookies be chosen?

- Example (Rosen p.338):

  How many solutions does the equation

  $x_1+x_2+x_3 = 11$

  have, where x_1, x_2, and x_3 are nonnegative integer?

Permutations with Indistinguishable Objects

- Example:

  How many different strings can be made by reordering the letters of the word

  PEPPERCORN

Permutations with Indistinguishable Objects

- The number of different ***permutations*** of ***n*** objects, where there are

  n_1 indistinguishable of type *1*,

  n_2 indistinguishable of type *2*,…, and

  n_k indistinguishable of type *k*,

  is:

$$\frac{n!}{n_1!n_2!\ldots n_k!}$$

Distributing Objects into Boxes

- Example (Rosen p.341):

  How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52?

Distributing Objects into Boxes

- The number of ways to distribute ***n*** distinguishable objects into k distinguishable boxes so that n_i objects are placed into box *i*, *i =1,2,…,k*, equals

$$\frac{n!}{n_1!n_2!\ldots n_k!}$$