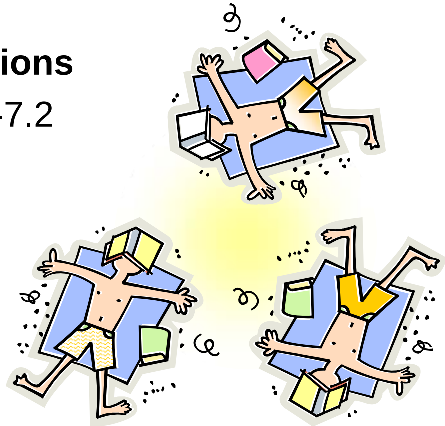
Recurrence Relations

- <u>Readings:</u>

  Recurrence Relations

  Rosen section 7.1-7.2



Recurrence Relations

- A ***recurrence relation*** for the sequence $\{a_n\}$ is an equation that <u>expresses a_n in terms of one or more of the previous terms</u>, $a_0, a_1, \ldots, a_{n-1}$.

$$a_n = 5a_{n-1}$$
$$b_n = b_{n-1} - 2\,b_{n-2} + 100$$
$$c_n = c_{n-3} + c_{n-4} + \log(n) + e^n$$

- A sequence is called a ***solution*** of a recurrence relation if its terms <u>satisfy the recurrence relation</u>.

Recurrence Relations

- <u>Example</u> (Rosen p.402):

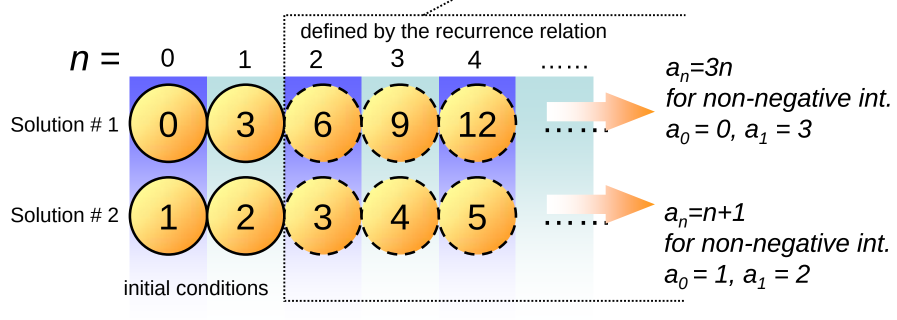
  Determine whether $a_n=3n$, for every nonnegative integer n, is a solution of

$$a_n = 2a_{n-1} - a_{n-2}\,; \quad n = 2, 3, \ldots$$

Initial Conditions

The ***initial conditions*** <u>specify the terms that precede the first term</u> where the recurrence relation takes effect.

$$a_n = 2a_{n-1} - a_{n-2}\,; \quad n = 2, 3, \ldots$$

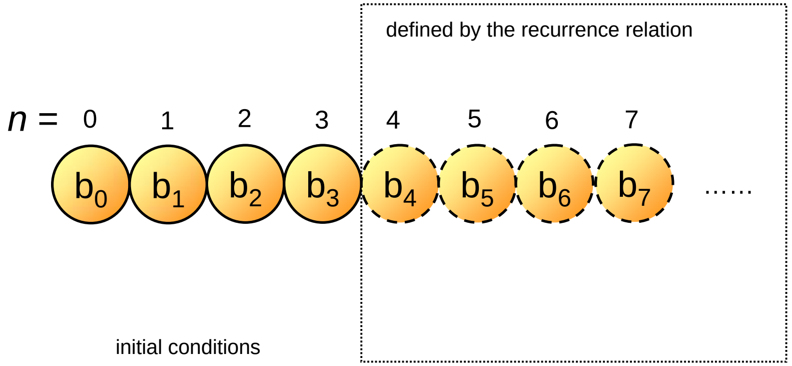


Initial Conditions

In order to find a *unique solution* for every non-negative integers to:

$$b_n = b_{n-2} + b_{n-4}\ ; \quad n = 4, 5, \ldots$$

how many terms of b_n needed to be given in the initial conditions?



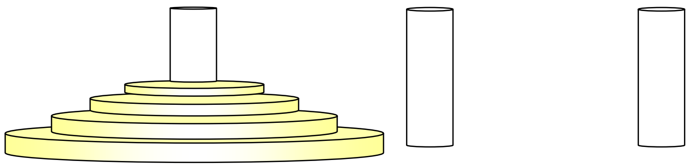
Modeling with Recurrence Relations

To find solutions for doing a task of a size *n*

Find a way to:
Construct the solution at the size n from the solution of the same tasks at smaller sizes.

Example: The Tower of Hanoi

Rules:
Move a disk at a time from one peg to another.
Never place a disk on a smaller disk.
The goal is to have all disk on the 2^{nd} peg in order of size.



Find H_n, the number of moves needed to solve the problem with *n* disks.

Example:

A man running up a staircase of *n* stairs. Each step he takes can cover either 1 or 2 stairs. How many different ways for him to ascend this staircase?

Solving Recurrence Relations

- A ***linear homogeneous*** recurrence relation can be solved in a systematic way.

<u>Linear homogeneous recurrence relation of degree *k* with constant coefficients</u>

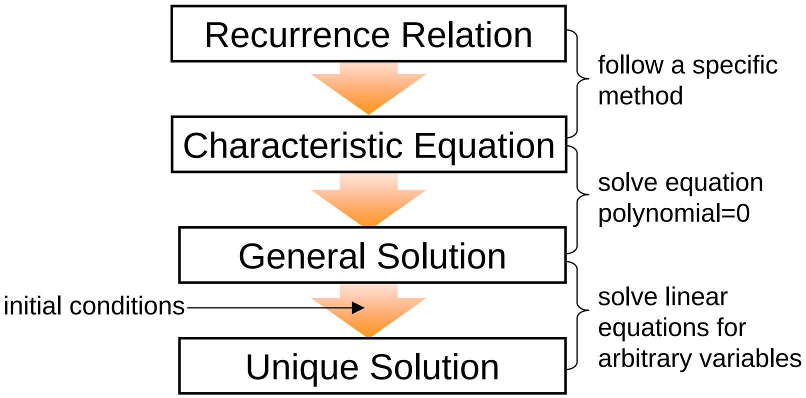
$$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$$

where $c_1, c_2, \ldots, c_k$ are real numbers and $c_k \neq 0$

Linear Homogeneous Recurrence Rel.

- $a_n=a_{n-1}+a^2_{n-2}$
- $H_n = 2H_{n-1}+1$
- $B_n = nB_{n-1}$
- $f_n = f_{n-1} + f_{n-2}$

Solving: Linear Homogeneous Recurrence Relations



Solving: Linear Homogeneous Recurrence Relations

- Recurrence Relation:
  $a_n - c_1a_{n-1} - c_2a_{n-2} - \ldots - c_ka_{n-k} = 0$
- Characteristic equation:
  $r^k - c_1r^{k-1} - c_2r^{k-2} \ldots - c^k = 0$

<u>Example</u>:

$a_n = a_{n-1}+3a_{n-2}$

$b_n = 2b_{n-1}-b_{n-2}+5b_{n-3}$

$d_n = d_{n-2}+d_{n-5}$

Solving: Linear Homogeneous Recurrence Relations

- Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose the ***characteristic equation***:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} \ldots - c^k = 0$$

has <u>k distinct roots</u> $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the ***recurrence relation***:

$$a_n - c_1 a_{n-1} - c_2 a_{n-2} - \ldots - c_k a_{n-k} = 0$$

if and only if:

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \ldots$. Where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

- <u>Example</u> :
  What is the solution of the recurrence relation:

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$?

- <u>Example</u>:
  What is the solution of the recurrence relation:

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with $a_0 = 2$, $a_1 = 5$ and $a_2 = 15$?

Repeated Roots

- Suppose the characteristic equation has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$.
- <u>Solution</u>:

$$\begin{aligned} a_n = & (\alpha_{1,0} + \alpha_{1,1} n + \ldots + \alpha_{1,m1-1} n^{m1-1}) r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1} n + \ldots + \alpha_{2,m2-1} n^{m2-1}) r_2^n \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + (\alpha_{t,0} + \alpha_{t,1} n + \ldots + \alpha_{t,mt-1} n^{mt-1}) r_t^n \end{aligned}$$

- Example :

  What is the solution of the recurrence relation:

  $$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$

  with $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$?

Solving: Linear Nonhomogeneous Recurrence Relations

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} + F(n)$$

Associated homogeneous recurrence relation

$\{a_n^h\}$ $\{a_n^p\}$

$$\{a_n\} = \{a_n^h\} + \{a_n^p\}$$

Solving: Linear Nonhomogeneous Recurrence Relations

- Key:

  1 – Solve for a solution of the associated homogeneous part.

  2 – Find a particular solution.

  3 – Sum the solutions in 1 and 2
- There is no general method for finding the particular solution for every $F(n)$
- There are general techniques for some $F(n)$ such as *polynomials* and *powers of constants*.

- Example:

  Find the solutions of $a_n = 3a_{n-1}+2n$ with $a_1 = 3$

- Example:

  Find the solutions of $a_n = 5a_{n-1} - 6a_{n-2}+7^n$

Particular Solutions

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \ldots + b_1 n + b_0)\, s^n$$

where $b_0, b_1, \ldots, b_t$ and s are real numbers.

When ***s*** is **not** a root of the characteristic equation:

The particular solution is of the form:

$$(p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0)\, s^n$$

When ***s*** is a root of multiplicity ***m***:

The particular solution is of the form:

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0)\, s^n$$

- Example:

  What form does a particular solution of

  $$a_n = 6a_{n-1} - 9a_{n-2} + F(n)$$

  have when:

  $F(n)= 3n$, $F(n)= n3^n$, $F(n)= n^2 2^n$, $F(n)= (n^2+1)3^n$?

- Example :

  Find the solution of $a_n = \sum_{k=1}^{n} k$