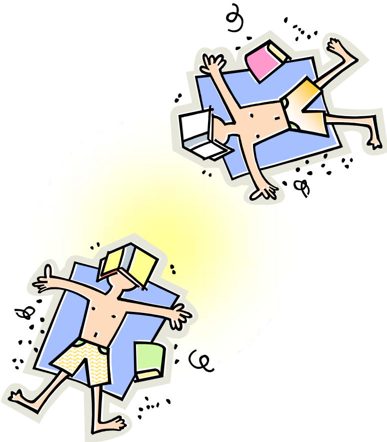
Generating Functions

- Readings:
  Generating Functions
  Rosen section 7.4



Generating Functions

- Used to **represent sequences** efficiently by coding the terms of a sequence **as coefficients of powers of a variable x in a power series**.

$a_o \quad a_1 \quad a_2 \quad a_3$

$a_n = 1, 2, 4, 8, \ldots. \quad \rightarrow G(x) = 1 + 2x + 4x^2 + \ldots$

$b_n = 1, -1, 1, -1, \ldots. \quad \rightarrow G(x) = 1 - x + x^2 - \ldots$

$b_o \quad b_1 \quad b_2 \quad b_3$

Generating Functions

- The generating function for the sequence $a_n = a_0, a_1, a_2, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{k=0}^{\infty} a_k x^k$$

- Example:
  Find the generating functions for the sequences $\{a_k\}$ with:

$a_k = 3$

$a_k = k + 1$

$a_k = 2^k$

$a_k = C(8,k)$

Why so?

- Generating functions can be used for:
  - solving many types of counting problems
  - solving recurrence relations
  - proving combinatorial identities

Useful Facts about Power Series

- $1/(1-x) = 1 + x + x^2 + \ldots$ for $|x| < 1$
- $1/(1-ax) = 1 + ax + ax^2 + \ldots$ for $|ax| < 1$

<u>Adding & multiplying two generating functions</u>

$$\text{Let } f(x) = \sum_{k=0}^{\infty} a_k x^k \text{ and } g(x) = \sum_{k=0}^{\infty} b_k x^k$$

$$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$$

$$f(x) g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j}\right) x^k$$

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- <u>Example</u> :

  Let $f(x) = \dfrac{1}{(1-x)^2}$. Find the coefficients in the expansion $f(x) = \sum_{k=0}^{\infty} a_k x^k$

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Extended Binomial Coefficient

- To apply binomial theorem for exponents that are not positive integers.

Let <u>*u* be a real number</u> and <u>*k* a nonnegative integer</u>. Then the extended binomial coefficient, $\binom{u}{k}$, is defined by:

$$\binom{u}{k} = \begin{cases} u(u-1)\ldots(u-k+1)/k! & \text{if } k > 0 \\ 1 & \text{if } k = 0 \end{cases}$$

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- Example:

$$\binom{-6}{3} =$$

$$\binom{0.8}{5} =$$

- When the top parameter is a negative number, the extended binomial coefficient can be expressed in terms of an ordinary binomial coefficient.

$$\binom{-n}{r} =$$

Extended Binomial Theorem

- Let *x* be a real number with $|x|<1$ and let u be a real number. Then

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$$

$$(1+x)^{-n} =$$

$$(1-x)^{-n} =$$

Counting Problems and Generating Functions

Example:

Find the number of solutions of $e_1+e_2+e_3=17$ where $2 \le e_1 \le 5$, $3 \le e_2 \le 6$, $4 \le e_3 \le 7$

- Example:

  Find the number of ways to insert tokens worth \$1, \$2, and \$5 into a vending machine to pay for an item that costs \$r:

  When order ***does not*** matter.

- Example:

  Find the number of ways to insert tokens worth \$1, \$2, and \$5 into a vending machine to pay for an item that costs \$r:

  When order ***does*** matter.

- Example:

  Use generating functions to find the number of *k-combination* of a set with *n* elements. (Assume that the binomial theorem has been established.)

- Example:

  Use generating functions to find the number of *r-combination* of a set with *n* elements when repetition of elements is allowed.

Using Generating Functions to Solve Recurrence Relations

- Example:

  $a_k = 3a_{k-1}$ for $k=1,2,3,\ldots$ and $a_0=2$

- Example:

  $a_n = 8a_{n-1} + 10^{n-1}$ and $a_0 = 1$

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Proving Identities via Generating Functions

- Example :

  Use generating functions to show that

$$\sum_{k=0}^{n} c(n,k)^2 = c(2n,n)$$

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