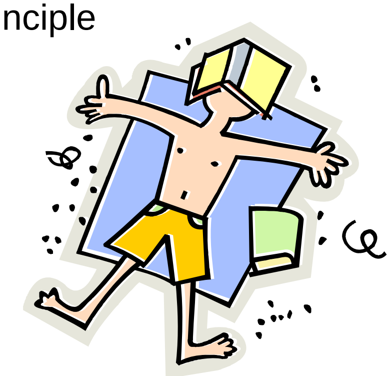
Counting Techniques

- Readings:

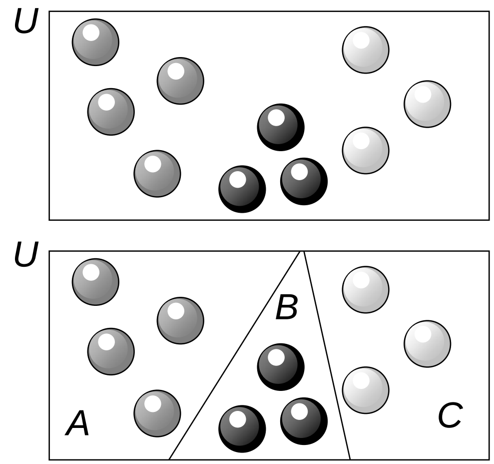
  5.1 The Basics of Counting

  5.2 The Pigeonhole Principle

  5.3 Permutations and Combinations



Counting by Cases



The Sum Rule

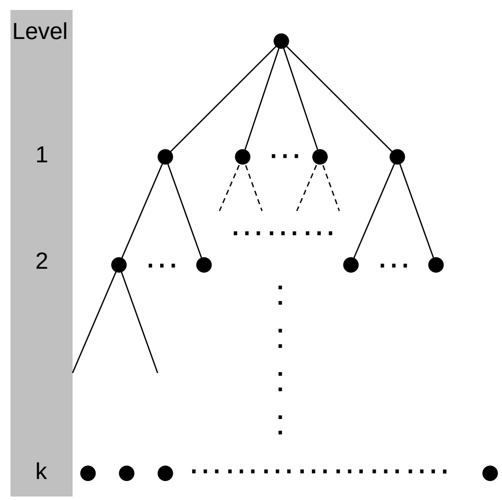
Tree Diagram

- help counting things composing of successive steps

E.g.: List all bit strings w/ 3 bits.

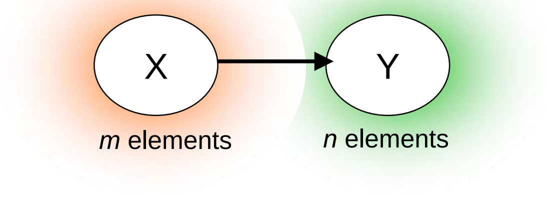
E.g.: List all bit strings w/ 3 bits but without ’00’

The Product Rule



Example

How many functions?



How many one-to-one function?

Basic Counting Principles

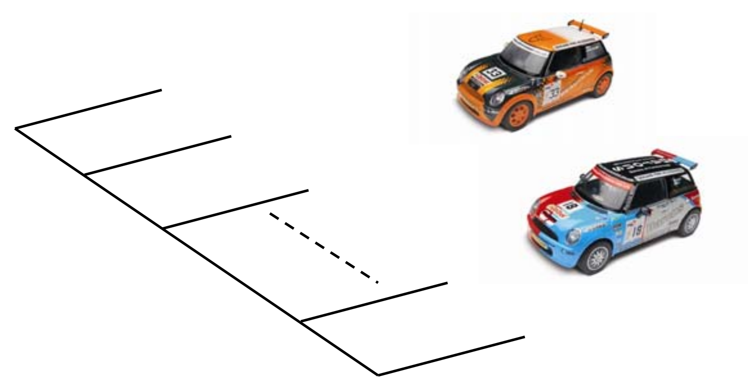
- Example : Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$

Example:

A password can contain 6 to 8 characters. Each character can be A-Z. How many possible passwords are there?

Example:

A parking lot consists of a single row of *n* parking spaces. Only two cars park in this parking lot. How many ways can they park?

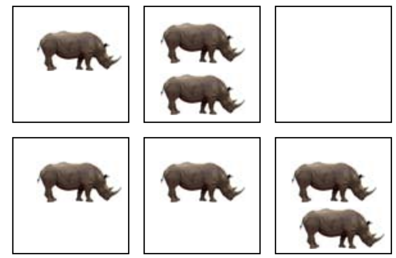


Example:

How many ways can they park if there can be at most one empty space between them?

The Pigeonhole Principle

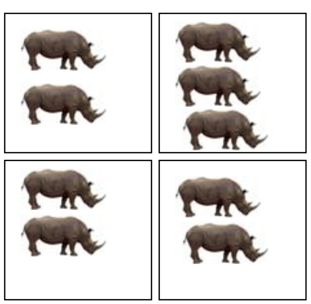
If *k+1* or more objects are placed into *k* boxes, then there are *at least one box containing two or more objects.*



6 boxes
7 objects

The Pigeonhole Principle

If *N* objects are placed into *k* boxes, then there is *at least one box containing at least* $\lceil N/k \rceil$ *objects.*

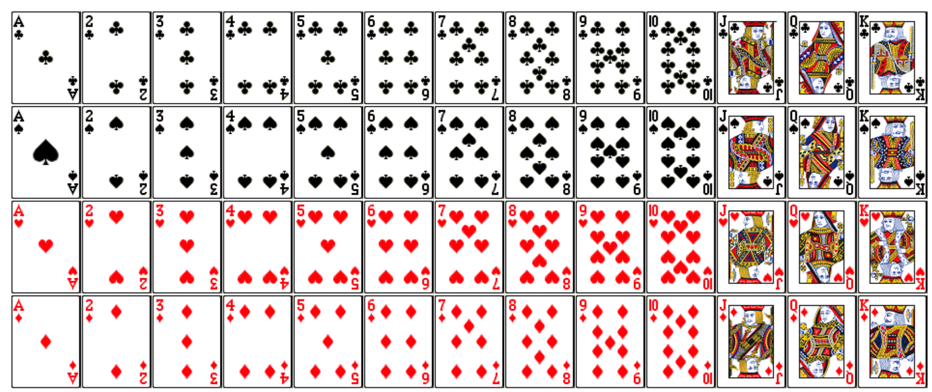


4 boxes
9 objects
$\lceil 9/4 \rceil = 3$

There is at least one box that contains at least 3 objects.

- Example:

  How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?



Permutations

- An <u>ordered</u> arrangement of *r* elements of a set is called an ***r-permutation***
- E.g.: S = {1,2,3}

  1,2 is a 2-permutation of S

  2,1 is another 2-permutation of S

  3,2 is also another 2-permutation of S

  1,2,3 is a permutation of S

  2,1,3 is another permutation of S

Permutations

The number of *r-permutations* of a set with *n* distinct elements is:

$$P(n,r) = n(n-1)(n-2)\ldots(n-r+1)$$

<u>Proof</u>:

- <u>Example</u>

  How many ways are there to select a 1^{st}-prize winner, a 2^{nd}-prize winner, and a 3^{rd}-prize winner from 100 people?

Combinations

- An ***r-combination*** of elements of a set is an unordered selection of r elements from the set.
- Or a subset, with r elements, of the set.

  E.g.: S = {1,2,3,4}

  {1,2,3} is a 3-combination of S

  {3,2,1} is the same as {1,2,3}

Combinations

The number of *r-combinations* of a set with n distinct elements is:

$C(n,r) = n! / r!(n-r)!$

Proof:

- Example:

  How many ways are there to select a 3 prize winners from 100 people (when the three prizes are identical)?

- Example:

  How many bit strings of length *10* contain more than 2 ones?

Example:

How many subsets of three different integers between 1 to 90 (inclusive) are there whose sum is an even number?

Permutations with Indistinguishable Objects

- Example:

  How many different strings can be made by reordering the string "*ABCDEFGHIJ*" ?

  How many different strings can be made by reordering the letters of the word

  "PEPPERCORN"

Permutations with Indistinguishable Objects

- The number of different ***permutations*** of n objects, where there are

  n_1 indistinguishable of type *1*,

  n_2 indistinguishable of type *2*,…, and

  n_k indistinguishable of type *k*,

  is:

$$\frac{n!}{n_1!n_2!...n_k!}$$

Distributing Objects into Boxes

- <u>Example</u>:

  How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52?

Distributing Objects into Boxes

- The number of ways to distribute ***n*** distinguishable objects into k distinguishable boxes so that n_i objects are placed into box *i*, *i =1,2,…,k*, equals

$$\frac{n!}{n_1!n_2!\ldots n_k!}$$