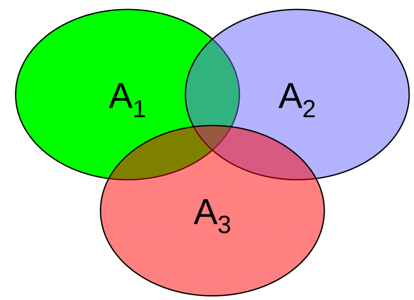
Inclusion-Exclusion Principle

- Readings:

  Rosen section 7.5-7.6

Inclusion-Exclusion

- How many elements are in the union of finite sets?



$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_3 \cap A_1| + |A_1 \cap A_2 \cap A_3|$$

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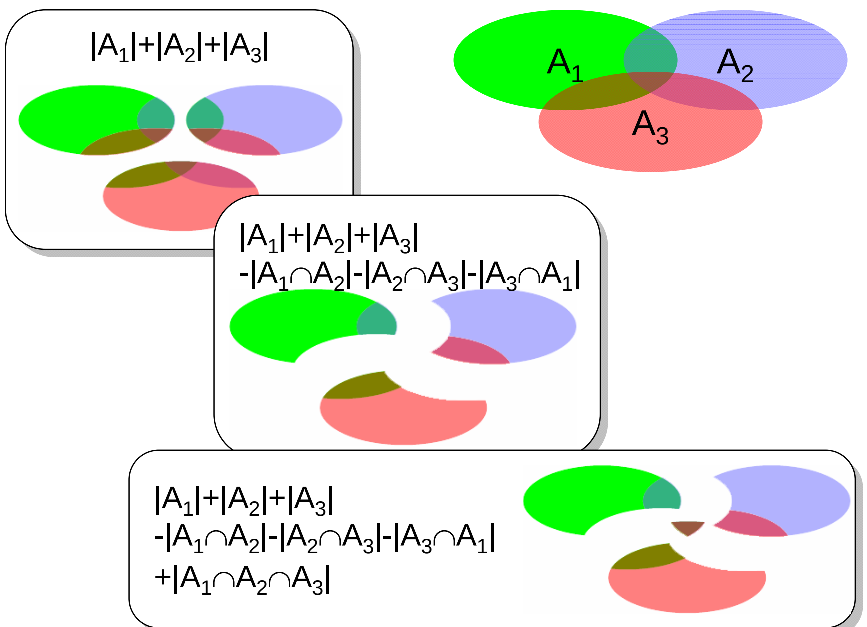

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The Principle of Inclusion-Exclusion

- Let $A_1, A_2, \ldots, A_n$ be finite sets. Then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i \le j \le n} |A_i \cap A_j| + \sum_{1 \le i \le j \le k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

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- Examples:

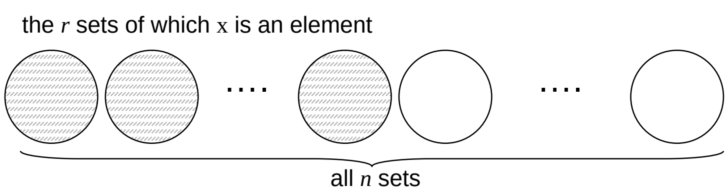
$$|A_1 \cup A_2 \cup A_3 \cup A_4| =$$

Proof: Inclusion-Exclusion Principle

- Showing that an element in the union is counted exactly once.

Let x be an element of exactly r sets.

For example, x is an element of $A_1, A_2, \ldots, A_r$,
But not of $A_{r+1}, A_{r+2}, \ldots, A_n$.



$$\sum_{1 \le i \le n} |A_i|$$

$$- \sum_{1 \le i1 \le i2 \le n} |A_{i1} \cap A_{i2}|$$

$$+ \sum_{1 \le i1 \le i2 \le i3 \le n} |A_{i1} \cap A_{i2} \cap A_{i3}|$$

$$- \cdots + (-1)^{r+1} \sum_{1 \le i1 \le i2 \le \cdots \le ir \le n} |A_{i1} \cap A_{i2} \cap \cdots \cap A_{ir}|$$

$$+ \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

Using the Formula to find $|A_1 \cap A_2 \cap \ldots \cap A_n|$

$$
\begin{aligned}
|A_1 \cap A_2 \cap \ldots \cap A_n| &= |U| - |(A_1 \cap A_2 \cap \ldots \cap A_n)'| \\
&= |U| - |A'_1 \cup A'_2 \cup \ldots \cup A'_n| \\
&= |U| - |B_1 \cup B_2 \cup \ldots \cup B_n|
\end{aligned}
$$

Let $B_i = A'_i$

Use the formula

Therefore, to find the number of elements in an intersection of sets

Another Notation

- To find elements with all properties $Q_1, Q_2, .., Q_n$
- Define properties $P_1, P_2, .., P_n$ *so that* P_i *is the opposite of* Q_i
- Let A_i be the subset of elements with property P_i.
- Let $N(P'_1 P'_2 \cdots P'_n)$ denote the number of elements with none of the properties $P_1, P_2, .., P_n$

$$N(Q_1 Q_2 \cdots Q_n) = N(P'_1 P'_2 \cdots P'_n) = N - |A_1 \cup A_2 \cup \cdots \cup A_n|$$

where N = the total number of elements.

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$$N(P'_1 P'_2 \cdots P'_n) = N - |A_1 \cup A_2 \cup \cdots \cup A_n|$$

$$
\begin{aligned}
N(P'_1 P'_2 \cdots P'_n) = N - (&\sum_{1 \le i \le n} |A_i| - \sum_{1 \le i \le j \le n} |A_i \cap A_j| \\
&+ \sum_{1 \le i \le j \le k \le n} |A_i \cap A_j \cap A_k| \\
&- \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|)
\end{aligned}
$$

$$
\begin{aligned}
N(P'_1 P'_2 \cdots P'_n) = N &- \sum_{1 \le i \le n} N(P_i) + \sum_{1 \le i \le j \le n} N(P_i P_j) \\
&- \sum_{1 \le i \le j \le k \le n} N(P_i P_j P_k) + \cdots + (-1)^n N(P_1 P_2 \cdots P_n)
\end{aligned}
$$

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- <u>Example</u>:

How many solutions does $x_1 + x_2 + x_3 = 11$ have,

where x_1 is a non negative integer ≤ 3,

x_2 is a non negative integer ≤ 4,

and x_3 is a non negative integer ≤ 6?

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The Number of Onto Functions

- Example:

  How many ways are there to assign five different jobs to four employees if every employee is assigned at least one job?

Derangements

- A ***derangement*** is a permutation of objects that leaves no object in its original position.
- Example:

  Consider a sequence 12345.

  21453

  43512

  42351

Derangements

- The number of derangements of a set with n elements, D_n = ?

- <u>Example</u>: “The Hatcheck Problem”

  An employee checks the hats of n people at a restaurant. He forgot to put claim check numbers on the hats. When customers return for their hats, this checker gives hats chosen at random to them.

  What is the probability that no one receives the correct hat?