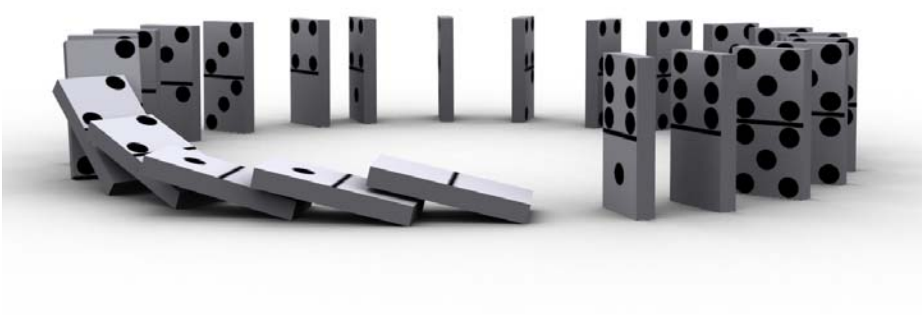
Mathematical Induction



Mathematical Induction

- A proof by induction that *P(n)* is true for every positive integer *n* consists of 2 steps:

  BASIC STEP: Show that *P(1)* is true.

  INDUCTIVE STEP:
  Show that $P(k) \rightarrow P(k+1)$ is true for every positive integer *k*

- Example :
  Prove that the sum of the first *n* odd positive integers is n^2.

  P(n):

  Basic Step:

  Inductive Step:

- Example:
  Prove that $n < 2^n$ for all positive integers *n*.

  P(n):

  Basic Step:

  Inductive Step:

- Example :

  Prove that n^3-n is divisible by 3 all positive integers n.

  P(n):

  Basic Step:

  Inductive Step:

Mathematical Induction

- Sometimes we want to prove that $P(n)$ is true for $n = b, b+1, b+2, \dots$ where b is an integer other than 1.

  BASIC STEP: Show that $P(b)$ is true.

  INDUCTIVE STEP:
  Show that $P(k) \rightarrow P(k+1)$ is true for every positive integer k

Example :

Prove that $H_{2^n} \geq 1 + \frac{n}{2}$ $\quad H_j = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}$

whenever n is a nonnegative integer.

P(n):

Basic Step:

Inductive Step:

Proving Mathematical Induction

- The well-ordering property:

Every nonempty set of nonnegative integers has a least element.

Proving Mathematical Induction

- Show that $P(n)$ must be true for all positive integers when $P(1)$ and $P(k) \rightarrow P(k+1)$ are true.
- Assume that $P(n)$ is not true for at least a positive integer. Then, the set S for which $P(n)$ is false is nonempty.
- S has the least element, called m. ($m \neq 1$)
- Since $m-1 < m$, then $m-1 \notin S$ (or $P(m-1)$ is true)
- But $P(m-1) \rightarrow P(m)$ is true. So, $P(m)$ must be true.
- This contradicts the choice of m.

Strong Induction

- A proof by induction that $P(n)$ is true for every positive integer n consists of 2 steps:
- Use a different induction step.

BASIC STEP: Show that $P(1)$ is true.

INDUCTIVE STEP:
Show that $[P(1) \wedge P(2) \wedge \ldots \wedge P(k)] \rightarrow P(k+1)$ is true for every positive integer k

- Example:
  Show that if n is an integer greater than 1, then n can be written as the product of primes.

P(n):

Basic Step:

Inductive Step:

- Example:
  Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.