Proof By Contradiction

and
other proof techniques

Proving p→q

From previous lecture

Direct Proof: Show that if **p** is true, **q** must be true.

Proof by Contraposition: $p \to q \equiv \neg q \to \neg p$

Show that if **¬q** is true, **¬p** must be true.

Universal instantiation / Universal generalization

From previous lecture

"If n is an even integer, n^2 is an even integer"

$$\forall_{n \in Z_{even}} (P(n) \to Q(n))$$

$P(n)$: n is an even integer.
$Q(n)$: n^2 is an even integer.

$\therefore P(c) \to Q(c)$ Universal Instantiation

proof → $P(c) \to Q(c) \equiv \mathrm{T}$
When c is any even integer

→ $\forall_{n \in Z_{even}} (P(n) \to Q(n)) \equiv \mathrm{T}$
Universal Generalization

<u>Example</u> (Rosen Ex.4 p.78):

Prove that if a and b are positive integers and $n = ab$, then $a \le \sqrt{n}$ or $b \le \sqrt{n}$

Proof by Contradiction

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- Proof by Contradiction
  - Suppose we want to prove a statement ***s***
  - Start by assuming **¬ *s*** is true.
  - Show that **¬ *s*** implies a contradiction. (¬ ***s*** → ***F***)
  - Then, **¬ *s*** must be false (or ***s*** must be true).

Proof by Contradiction

- Example:
  Show that at least 10 of any 64 days chosen must fall on the same day of the week.

Proof p→q by Contradiction

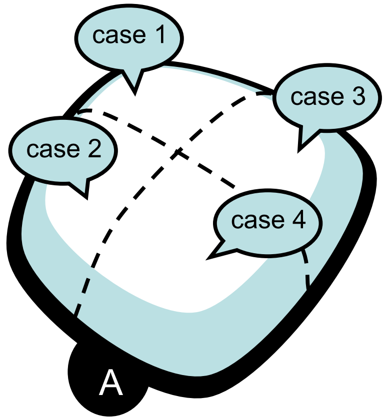
- Proof by Contradiction
  - Start by assuming **¬ *(p→q)*** is true.
  - That means ***p* ∧ ¬ *q*** is true.
    (since $\neg(p \rightarrow q) \equiv \neg(\neg p \vee q) \equiv p \wedge \neg q$)
  - Show that ***p* ∧ ¬ *q*** is a contradiction
  - Then, **¬ *(p→q)*** must be false
    (or ***(p→q)*** must be true).

Proving p→q

- Example:
  Prove that "If *n* is an integer and n^3+5 is odd, then *n* is even". Using a proof by contradiction.

Proof by Cases

$$A = CASE1 \cup CASE2 \cup CASE3 \cup CASE4$$



$\forall_{n \in CASE1}(P(n) \to Q(n))$

$\forall_{n \in CASE2}(P(n) \to Q(n))$

$\forall_{n \in CASE3}(P(n) \to Q(n))$

$\forall_{n \in CASE4}(P(n) \to Q(n))$

$\forall_{n \in A}(P(n) \to Q(n))$

Proof by Cases

- Example:
  Show that |*xy*| = |*x*||*y*|, where *x* and *y* are real numbers.

Proof of p↔q

- Since (p↔q) ↔ (p→q) ∧ (q→p), then *prove both p→q and q→p*
- Equivalent propositions ($p_1 \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_n$) are proven by *proving* $p_1 \to p_2$, $p_2 \to p_3$, …, $p_n \to p_1$

Equivalent Propositions

- Example
  Show that these statements are equivalent:
  p_1: *n* is an even integer.
  p_2: *n -1* is an odd integer.
  p_3: n^2 is an even integer.

Proof of Proposition Involving Quantifiers

- Existence proofs: A proof of $\exists x P(x)$
- *Constructive existence proof:*
  - Find an element *c* such that P(*c*) is true.
- *Non-constructive existence proof:*
  - Do not find an element *c* such that P(*c*) is true, but use some other ways.

Existence Proofs

- Example :

  Show that $\exists x \exists y$ (x^y is rational.) where x and y are irrational.

Proof of Proposition Involving Quantifiers

- Uniqueness proofs: showing that there is a unique element x such that P(x).

  1) *Existence*:

     Show that $\exists x P(x)$

  2) *Uniqueness*:

     Show that if $y \neq x$, P(y) is false.

- is the same as proving:

$$\exists x(P(x) \wedge \forall y(y \neq x \rightarrow \neg P(y)))$$

Uniqueness Proofs

- Example:

  Show every integer has a unique additive inverse. (If p is an integer, there exists a unique integer q such that p+q = 0.)

Counterexamples

- Show that ∀xP(x) is false.

- Example:
  "Every positive integer is the sum of the squares of three integers" ?