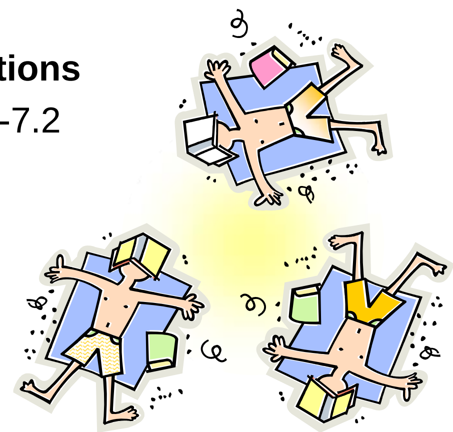
Recurrence Relations

- Readings:
  Recurrence Relations
  Rosen section 7.1-7.2



Recurrence Relations

- A ***recurrence relation*** for the sequence $\{a_n\}$ is an equation that <u>expresses a_n in terms of one or more of the previous terms</u>, $a_0, a_1, \ldots, a_{n-1}$.

$$a_n = 5a_{n-1}$$
$$b_n = b_{n-1} - 2\, b_{n-2} + 100$$
$$c_n = c_{n-3} + c_{n-4} + \log(n) + e^n$$

- A sequence is called a ***solution*** of a recurrence relation if its terms <u>satisfy the recurrence relation</u>.

Recurrence Relations

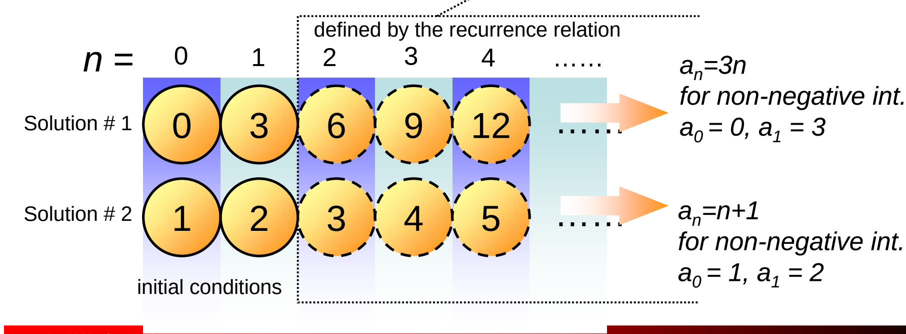
- <u>Example</u>
  Determine whether $a_n=3n$, for every nonnegative integer n, is a solution of

$$a_n = 2a_{n-1} - a_{n-2}\,; \quad n = 2, 3, \ldots$$

Initial Conditions

The ***initial conditions*** <u>specify the terms that precede the first term</u> where the recurrence relation takes effect.

$$a_n = 2a_{n-1} - a_{n-2}\,; \quad n = 2, 3, \ldots$$

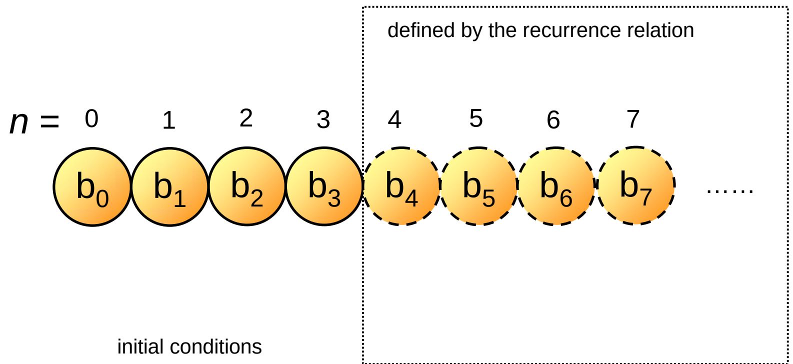


Initial Conditions

In order to find a *unique solution* for every non-negative integers to:

$$b_n = b_{n-2}+b_{n-4}\ ; \quad n = 4, 5, \ldots$$

how many terms of b_n needed to be given in the initial conditions?



Modeling with Recurrence Relations

To find solutions for doing a task of a size *n*

Find a way to:

Construct the solution at the size n from the solution of the same tasks at smaller sizes.

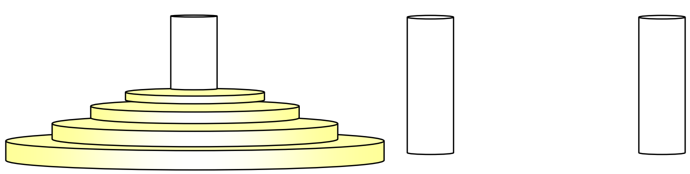
Example: The Tower of Hanoi

Rules:

Move a disk at a time from one peg to another.

Never place a disk on a smaller disk.

The goal is to have all disk on the 2^{nd} peg in order of size.



Find H_n, the number of moves needed to solve the problem with *n* disks.

Example:

A man running up a staircase of n stairs. Each step he takes can cover either 1 or 2 stairs. How many different ways for him to ascend this staircase?

Linear Recurrence Relations with Constant Coefficients

Homogeneous

$a_n = c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k}$

Non-homogeneous

$a_n = c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k} + F(n)$

where $c_1, c_2, \dots, c_k$ are real numbers and $c_k \neq 0$

Degree = k

Linear Homogeneous Recurrence Rel.

- $a_n=a_{n-1}+a^2_{n-2}$
- $H_n = 2H_{n-1}+1$
- $B_n = nB_{n-1}$
- $f_n = f_{n-1} + f_{n-2}$

Solving Recurrence Relations

- A ***linear homogeneous*** recurrence relation can be solved in a systematic way.

Some Solutions

Show that:

$$a_n = K_1 r^n$$

where:

K_1 can be any real number and,
we can choose the value of r to be anything.

is a solution of:

$$a_n = c_1 a_{n-1} + c_1 a_{n-2} + \ldots + c_k a_{n-k}$$

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Some Solutions

Show that:

$$a_n = K_1 r_1^n + K_2 r_2^n + \ldots + K_k r_k^n$$

where:

K_i can be any real number,
each r_i is a root of $r^k = c_1 r^{k-1} + c_2 r^{k-2} + \ldots + c_k$
and there are k distinct r_i's.

is a solution of:

$$a_n = c_1 a_{n-1} + c_1 a_{n-2} + \ldots + c_k a_{n-k}$$

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So far, we have found that ...

$$a_n = c_1a_{n-1} + c_1a_{n-2} + \ldots + c_ka_{n-k}$$

has “a” solution in the form of

$$a_n = K_1r_1^n + K_2r_2^n + \ldots + K_kr_k^n$$

where all r_i’s are the distinct roots of $r^k = c_1r^{k-1} + c_2r^{k-2} + \ldots + c_k$

Characteristic Equation

? Do all solutions have to be in this form?

Prove later!

Unique Solution

$$a_n = K_1r_1^n + K_2r_2^n + \ldots + K_kr_k^n$$

Without fixing the constants, they are not unique.

E.g.: $a_n = K_1(2)^n + K_2(3)^n$

$a_n = 2(2)^n + 1(3)^n \rightarrow$ 3, 7, 17, …

$a_n = 1(2)^n + 2(3)^n \rightarrow$ 3, 8, 22, …

$a_n = 3(2)^n + 3(3)^n \rightarrow$ 6, 15, 39, …

Must fix the initial conditions

Find the solution of

$a_n = -5a_{n-1} - 6a_{n-2}$ $\quad n = 2,3,4,\ldots$

if $a_0 = 3, a_1 = 7$

Next Step

- Prove that all solutions must be in the form we have just shown in this lecture.
- What if the characteristic equation of the k order have less than k distinct roots?
- How to solve a linear non-homogeneous recurrence relation?