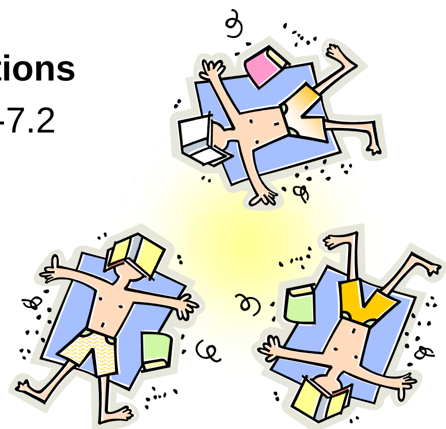
Recurrence Relations (Continue)

- <u>Readings:</u>
  Recurrence Relations
  Rosen section 7.1-7.2



Solutions to Linear Homogeneous Recurrence Relations

Sufficient Condition

Any sequences in the form: $a_n = K_1 r_1^n + K_2 r_2^n + \ldots + K_k r_k^n$

is a solution to: $a_n = c_1 a_{n-1} + c_1 a_{n-2} + \ldots + c_k a_{n-k}$

Necessary Condition

All solutions to: $a_n = c_1 a_{n-1} + c_1 a_{n-2} + \ldots + c_k a_{n-k}$

must be in the form: $a_n = K_1 r_1^n + K_2 r_2^n + \ldots + K_k r_k^n$

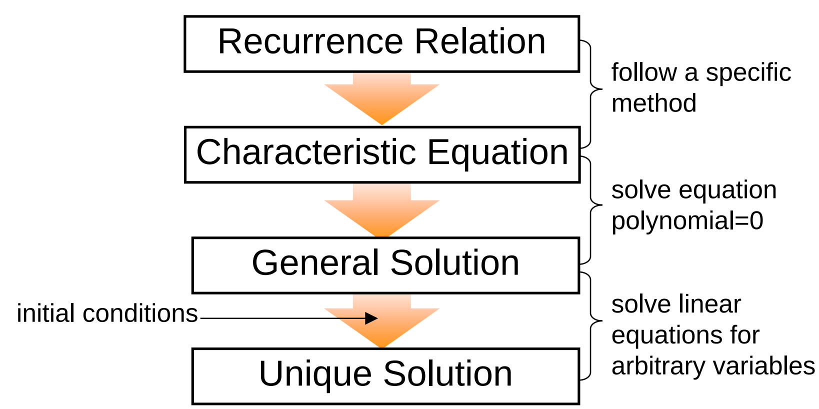
when the characteristic equation of the recurrence relation

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \ldots + c_k$$

has k distinct roots which are $r_1, r_2, \ldots, r_k$

Proof of the Necessary Condition

Solving: Linear Homogeneous Recurrence Relations



- Example:

  What is the solution of the recurrence relation:

  $$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

  with $a_0 = 2$, $a_1 = 5$ and $a_2 = 15$?

Repeated Roots

- Suppose the characteristic equation has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$.
- Solution:

$$\begin{aligned} a_n = & (\alpha_{1,0} + \alpha_{1,1}n + \ldots + \alpha_{1,m1-1}n^{m1-1})r_1^n \\ & +(\alpha_{2,0} + \alpha_{2,1}n + \ldots + \alpha_{2,m2-1}n^{m2-1})r_2^n \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & +(\alpha_{t,0} + \alpha_{t,1}n + \ldots + \alpha_{t,mt-1}n^{mt-1})r_t^n \end{aligned}$$

- Example :

  What is the solution of the recurrence relation:

  $$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$

  with $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$?

Solving: Linear Nonhomogeneous Recurrence Relations

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} + F(n)$$

Associated homogeneous recurrence relation

$\{a_n^h\}$ $\{a_n^p\}$

$$\{a_n\} = \{a_n^h\} + \{a_n^p\}$$

Solving: Linear Nonhomogeneous Recurrence Relations

- Key:

  1 – Solve for a solution of the associated homogeneous part.

  2 – Find a particular solution.

  3 – Sum the solutions in 1 and 2
- There is no general method for finding the particular solution for every $F(n)$
- There are general techniques for some $F(n)$ such as *polynomials* and *powers of constants*.

Particular Solutions

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \ldots + b_1 n + b_0)\, s^n$$

where $b_0, b_1, \ldots, b_t$ and s are real numbers.

When ***s*** is **not** a root of the characteristic equation:

The particular solution is of the form:

$$(p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0)\, s^n$$

When ***s*** is a root of multiplicity ***m***:

The particular solution is of the form:

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0)\, s^n$$

- Example:

  Find the solutions of $a_n = 3a_{n-1}+2n$ with $a_1 = 3$

- Example:

  Find the solutions of $a_n = 5a_{n-1} - 6a_{n-2}+7^n$

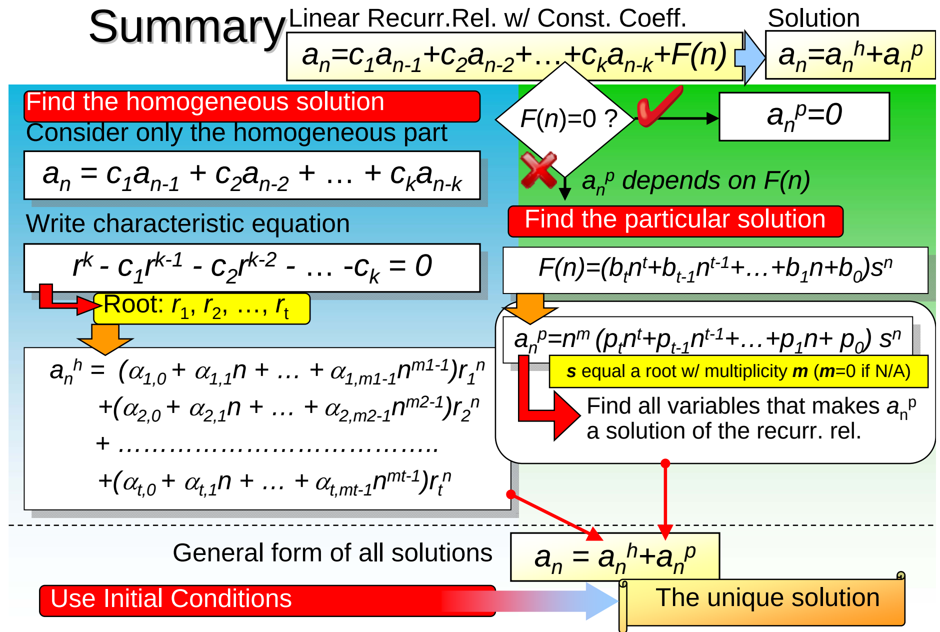
- Example:
  What form does a particular solution of

$$a_n = 6a_{n-1} - 9a_{n-2} + F(n)$$

  have when:

  $F(n)= 3n$, $F(n)= n3^n$, $F(n)= n^2 2^n$, $F(n)= (n^2+1)3^n$?

Summary



- Example :
  Find the solution of $a_n = \sum_{k=1}^{n} k$