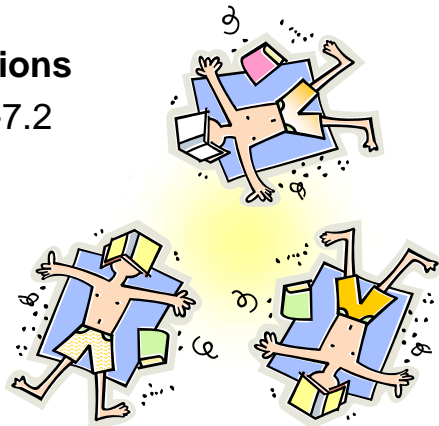


Recurrence Relations (Continue)

- Readings:
Recurrence Relations
 Rosen section 7.1-7.2



Solutions to Linear Homogeneous Recurrence Relations

Sufficient Condition

Any sequences in the form: $a_n = K_1 r_1^n + K_2 r_2^n + \dots + K_k r_k^n$
 is a solution to: $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$

Necessary Condition

All solutions to: $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$
 must be in the form: $a_n = K_1 r_1^n + K_2 r_2^n + \dots + K_k r_k^n$

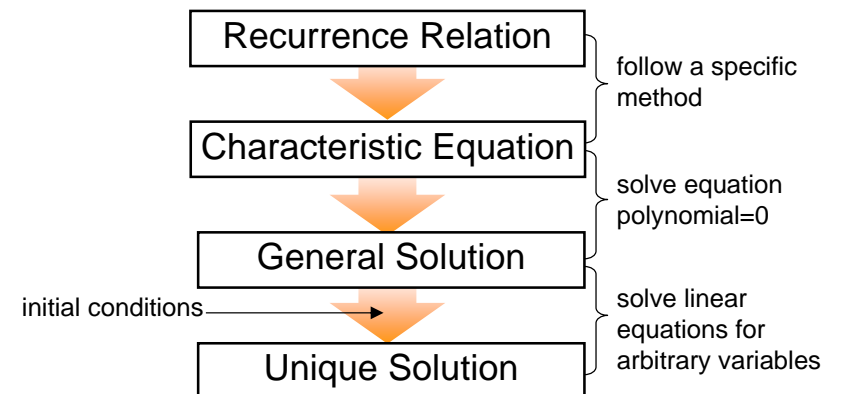
when the characteristic equation of the recurrence relation

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \dots + c_k$$

has k distinct roots which are r_1, r_2, \dots, r_k

Proof of the Necessary Condition

Solving: Linear Homogeneous Recurrence Relations



- Example:

What is the solution of the recurrence relation:

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with $a_0 = 2$, $a_1 = 5$ and $a_2 = 15$?

Repeated Roots

- Suppose the characteristic equation has t distinct roots r_1, r_2, \dots, r_t with multiplicities m_1, m_2, \dots, m_t
- Solution:

$$a_n = (\alpha_{1,0} + \alpha_{1,1}n + \dots + \alpha_{1,m_1-1}n^{m_1-1})r_1^n + (\alpha_{2,0} + \alpha_{2,1}n + \dots + \alpha_{2,m_2-1}n^{m_2-1})r_2^n + \dots + (\alpha_{t,0} + \alpha_{t,1}n + \dots + \alpha_{t,m_t-1}n^{m_t-1})r_t^n$$

- Example :

What is the solution of the recurrence relation:

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$

with $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$?

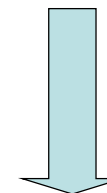
Solving: Linear Nonhomogeneous Recurrence Relations

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k} + F(n)$$

Associated homogeneous recurrence relation



$$\{a_n^h\}$$



$$\{a_n^p\}$$

$$\{a_n\} = \{a_n^h\} + \{a_n^p\}$$

Solving: Linear Nonhomogeneous Recurrence Relations

- Key:
 - 1 – Solve for a solution of the associated homogeneous part.
 - 2 – Find a particular solution.
 - 3 – Sum the solutions in 1 and 2
- There is no general method for finding the particular solution for every $F(n)$
- There are general techniques for some $F(n)$ such as *polynomials* and *powers of constants*.

Particular Solutions

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \dots + b_1 n + b_0) s^n$$

where b_0, b_1, \dots, b_t and s are real numbers.

When s is **not** a root of the characteristic equation:

The particular solution is of the form:

$$(p_t n^t + p_{t-1} n^{t-1} + \dots + p_1 n + p_0) s^n$$

When s is a root of multiplicity m :

The particular solution is of the form:

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \dots + p_1 n + p_0) s^n$$

- Example:
Find the solutions of $a_n = 3a_{n-1} + 2n$ with $a_1 = 3$

- Example:
Find the solutions of $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$

• Example:

What form does a particular solution of

$$a_n = 6a_{n-1} - 9a_{n-2} + F(n)$$

have when:

$$F(n) = 3n, F(n) = n3^n, F(n) = n^22^n, F(n) = (n^2+1)3^n?$$

Summary

Linear Recurr.Rel. w/ Const. Coeff.

Solution

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n) \rightarrow a_n = a_n^h + a_n^p$$

Find the homogeneous solution
Consider only the homogeneous part

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

Write characteristic equation

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_k = 0$$

Root: r_1, r_2, \dots, r_t

$$a_n^h = (\alpha_{1,0} + \alpha_{1,1}n + \dots + \alpha_{1,m_1-1}n^{m_1-1})r_1^n + (\alpha_{2,0} + \alpha_{2,1}n + \dots + \alpha_{2,m_2-1}n^{m_2-1})r_2^n + \dots + (\alpha_{t,0} + \alpha_{t,1}n + \dots + \alpha_{t,m_t-1}n^{m_t-1})r_t^n$$

$F(n)=0$?

$$a_n^p = 0$$

a_n^p depends on $F(n)$

Find the particular solution

$$F(n) = (b_1 n^t + b_{t-1} n^{t-1} + \dots + b_1 n + b_0) s^n$$

$$a_n^p = n^m (p_1 n^t + p_{t-1} n^{t-1} + \dots + p_1 n + p_0) s^n$$

s equal a root w/ multiplicity m ($m=0$ if N/A)

Find all variables that makes a_n^p a solution of the recurr. rel.

General form of all solutions

$$a_n = a_n^h + a_n^p$$

Use Initial Conditions

The unique solution

• Example :

Find the solution of
$$a_n = \sum_{k=1}^n k$$