A Sufficient Condition for Capturing an Object in the Plane with Disc-Shaped Robots

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Abstract: *An object is captured when it is restricted to stay within a bounded region of the workspace. In this paper, we present a sufficient condition for a team of disc-shaped robots to capture a two dimensional rigid object in the plane by enclosing it in a capturing formation. This condition is defined in terms of the robots' positions and a certain geometric property of the object. We do not assume any particular geometry, therefore the condition immediately holds for all object shapes. We also sketch an application of the capturing formation to the problem of object manipulation.*

1 Introduction

An object is captured when it is restricted to stay within a bounded region of the workspace, i.e., there exists no trajectory to bring the object to infinity. In this paper, we are interested in constraining (not necessarily immobilizing) a rigid two-dimensional object to stay within the midst of a team of robots. In particular, we present a sufficient condition for a team of disc-shaped robots to capture a given object in the plane by enclosing it in a cycle formation[1]. This capturing condition is defined in terms of the distance between consecutive robots in the cycle and a certain geometric property derived from the boundary of the object. The condition does not assume any particular geometry, therefore it immediately holds for all shapes of the object.

A capturing action generally applies to a set of the object's configurations rather than a single one. It provides a means with which uncertainty in the object's configuration can be handled.[2] However, the problem of capturing objects has so far received little attention in robotics. It was introduced in [11] the concept of Inescapable Configuration Space (ICS) region, i.e., on the idea of characterizing the regions of configuration space in which the object is not immobilized but is constrained to lie within a bounded region of the free configuration space (see [9] for similar work in the two-finger case). This concept is used in [10] as a basis for computing a plan for manipulating polygonal objects using three disc-shaped robots. Although the contact dynamics can be safely ignored, polygonal object model is assumed and the resulting plans may contain many very short steps because the ICS region is often very small due to the computation that takes into account only three chosen edges. The ICS is defined in the combined object/robot configuration space while our capturing concept is purely derived in the workspace of the object. This results in a much simpler capturing condition than that of ICS and allows the entire boundary of the object to be taken into account.

As a secondary purpose of the paper, we show how to apply the concept of capturing formation to the problem of nonprehensible object manipulation. We apply cooperative pushing by a team of robots. The motivation comes from the difficulty in handling an object when it is too large to be grasped by a single robot, or when the available robot is not equipped with grasping capability. Influenced by [6], pushing has been recognized as a useful process in object manipulation [1, 5, 7]. The common approach of the works rely on the Coulomb friction model and the quasi-static assumption. Based on contact dynamics modeling which is a priori unverifiable, the assumption limits motion of the pusher to be slow enough that inertial forces can be ignored and requires that the contact between the pusher and the object be maintained during the entire manipulation. Avoiding these shortcomings, our approach to object manipulation bypasses the need for contact dynamics modeling by taking advantage of the ability to kinematically prevent the object from escaping. This is accomplished by computing trajectories of the robots such that the corresponding formation can always capture the object (as if the object is transported in a moving cage). The overall idea is similar to that of [10] except that our approach, as mentioned earlier, can apply to objects other than polygons and extra robots may be easily added to a manipulation task for improving the robustness.

The rest of the paper is organized as follows. In section 2, we will present a sufficient condition of the capturing formation for convex objects. The condition is proven in

[1] arrangement of robots in a simple loop.

[2] For example, to ensure a successful grasp it is desirable to capture the object before grasping especially when all the contacts may not be made simultaneously during the grasp execution (because the object may move away from the initial configuration for which the grasp is computed).

Lemma 2 using heavily convexity of the captured objects. In Section 3, Lemma 2 is extended to construct a sufficient condition for capturing nonconvex objects. We will also sketch how a capturing formation can be generated. Then in Section 4 we discuss how capturing formation may be used as a framework for object manipulation. We complete the paper with conclusion in Section 5.

2 Capturing Formation

A team of robots form a capturing formation for an object when the object is restricted by the formation to stay within a bounded region of the workspace. The objective of this section is to present a sufficient condition of a capturing formation for a given convex object. This condition will be extended to handle nonconvex objects in the next section. We show in Lemma 2 that a convex object can be captured by enclosing it in a cycle formation of robots such that the distance between every pair of consecutive robots in the cycle is smaller than the *width* of the convex object. We will formally define the width and then prove Lemma 1 which is the main foundation of the work in this paper.

Let us consider a convex object $\mathcal{B}$ and assume that it has a C^1-continuous[3] boundary (this assumption is only for the convenience in proving Lemma 1; without loss of generality, we will later show that it can be safely removed).

Definition 1 *For a fixed orientation of $\mathcal{B}$, the parallel envelop $\mathcal{E}(\theta)$ is defined to be a pair of the closest parallel lines such that the angle between the lines and the x-axis is θ and the region bounded by the two lines contains the convex object $\mathcal{B}$ (see Figure 1). Also, let $d_{\mathcal{E}} : S^1 \mapsto \boldsymbol{R}$ be a function mapping an angle θ to the shortest distance between the two parallel lines of the envelop $\mathcal{E}(\theta)$.*

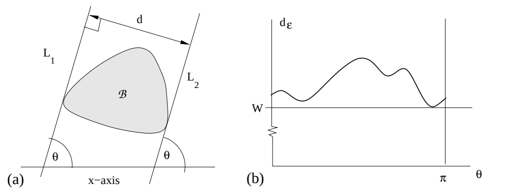

Figure 1: (a) Parallel lines L_1 and L_2 touch the convex $\mathcal{B}$ and form the parallel envelop $\mathcal{E}(\theta)$ with $d_{\mathcal{E}}(\theta) = d$, and (b) the corresponding function $d_{\mathcal{E}}$ in θ.

We follow [3] to call $d_{\mathcal{E}}$ as the diameter function. In that paper, the diameter function is used for computing the squeeze function for orientation planning of polygons using a frictionless gripper without sensors. The paper also presents an $O(n)$ algorithm for computing the diameter function of n-gon figures.

[3]A parametric curve $r(t)$ is C^k-continuous (or k-smooth) if all derivatives of the curve exists up to the kth order.

The following definition formally define the width of a convex object.

Definition 2 *The width of $\mathcal{B}$, denoted hereafter by W, is the minimum of the diameter function $d_{\mathcal{E}}(\theta)$ for all angles $\theta \in S^1$.*

The following lemma is the key foundation of the work presented here. Let us imagine two rooms separated by an infinitely long straight wall with one open door. The lemma essentially states that a convex object cannot go from one room to the other if the door is narrower than the width of the object. The proof of the lemma relies heavily on convexity of the object. It traces the two inward normals at the two intersection points between the object's boundary and a fixed vertical line as the object moves from one side of the line to the other. The proof shows that no matter which trajectory is chosen, the object always, at a certain moment, intersects the vertical line in a segment that is not smaller than the width of the object.

Lemma 1 *Let G be a vertical line with a gap of length d. If $d < W$, no trajectory can bring the convex object $\mathcal{B}$ from being entirely in one half plane completely to the other without colliding with G (the two half planes are separated by the line supporting G; see Figure 5(a)).*

PROOF: Let L be a fixed vertical line and $\boldsymbol{e} = (0,1)^T$ be a unit vector pointing upward (Figure 2). By convexity, a convex body intersects line L in a line segment (or a point when they only touch). This line segment is bounded by two endpoints: the upper endpoint $\boldsymbol{p}_u$ and the lower endpoint $\boldsymbol{p}_l$. Because an inward normal must point toward the half plane containing the convex body (and, of course, the intersection line segment), it is clear that

$$\boldsymbol{n}_u \cdot \boldsymbol{e} \leq 0 \text{ and } \boldsymbol{n}_l \cdot \boldsymbol{e} \geq 0, \qquad (1)$$

where $\boldsymbol{n}_u$ and $\boldsymbol{n}_l$ are, respectively, the inward normals of the convex body's boundary at the upper and lower endpoints (Figure 2(b)).

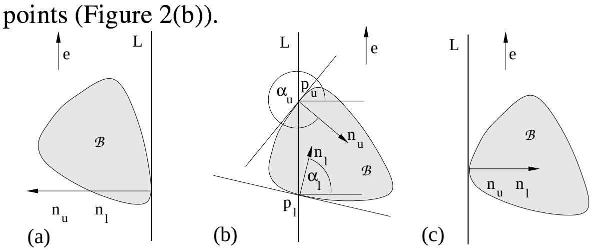

Figure 2: The convex $\mathcal{B}$ moves from the left to the right half plane in three steps (see text).

Let α_u and α_l be the angles between $\boldsymbol{n}_u$ and $\boldsymbol{n}_l$ with the x-axis. Now let us consider the inward normals $\boldsymbol{n}_u$ and $\boldsymbol{n}_l$ as $\mathcal{B}$ moves from being entirely in the left half plane completely to the right half plane. Regardless of the trajectory taken, it is obvious that successful passage must contain the following three sequential steps: (1) $\mathcal{B}$ is entirely in the left half plane and touches line L from the left(Figure 2(a)), (2) $\mathcal{B}$ intersects L in a line segment (Figure 2(b)), and (3) $\mathcal{B}$ has just completely moved into the right half plane and touches L from the right (Figure 2(c)). In step one, we have $\alpha_l = \alpha_u = \pi$. From Inequalities 1, α_l varies in the range $[0, \pi]$ during step two, and it reaches 0 in step three. Likewise, α_u varies in the range $[\pi, 2\pi]$ during step two, and it reaches 2π in step three. We can see that $0 \leq \alpha_u - \alpha_l \leq 2\pi$ and the difference $\alpha_u - \alpha_l$ goes from 0 in step one to 2π in step three. Therefore, by continuity, at a certain moment, it holds that $\alpha_u - \alpha_l = \pi$ which corresponds to having two parallel tangents at the endpoints $\boldsymbol{p}_u$ and $\boldsymbol{p}_l$ (Figure 3).

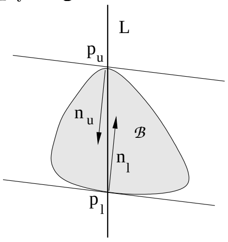

Figure 3: At a certain moment, the tangents at the endpoints $\boldsymbol{p}_u$ and $\boldsymbol{p}_l$ must become parallel.

With an appropriate rotation and proper angle α, the two parallel tangents form the parallel envelop $\mathcal{E}(\alpha)$. Because $|\boldsymbol{p}_u - \boldsymbol{p}_l| \geq d_{\mathcal{E}}(\alpha)$ and $d_{\mathcal{E}}(\alpha) \geq W$, we have $|\boldsymbol{p}_u - \boldsymbol{p}_l| \geq W$. This means that at this moment, the object $\mathcal{B}$ must intersect line L in a line segment that is not smaller than W. In other words, assuming that G is supported by L, if the gap of G is smaller than W, the convex object $\mathcal{B}$ cannot successfully move from the left half plane to the right half plane without colliding with G. ■

Note that Lemma 1 still holds without the assumption that the boundary of $\mathcal{B}$ be $\mathcal{C}^1$-continuous. This is because even when the boundary of $\mathcal{B}$ is only $\mathcal{C}^0$-continuous, as illustrated in Figure 4, it is clear that we can always find a convex subset $\mathcal{B}' \subset \mathcal{B}$ with a $\mathcal{C}^1$-continuous boundary ($\mathcal{B}'$ can also be chosen to be an arbitrarily close approximation of $\mathcal{B}$). In this case, because Lemma 1 certainly applies for $\mathcal{B}'$, it can be deduced from rigidity of $\mathcal{B}$ that $\mathcal{B}$ cannot cross from one half plane to the other. In other words, Lemma 1 requires only that $\mathcal{B}$ has a closed boundary.

A wall with a gap that is narrow enough can constrain an object in one room from moving completely to the other. The following definition defines a term for referring to the room (the half plane) in which the object is confined. This definition is used in Lemma 2, which extends Lemma 1 to provide a sufficient condition for a cycle formation of the robots to be able to capture the convex object $\mathcal{B}$. From now on, we denote by $\mathcal{B}(\boldsymbol{q})$ the plane region occupied by $\mathcal{B}$ when it is at the configuration $\boldsymbol{q}$.

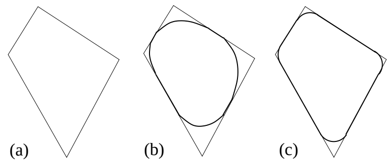

Figure 4: (a) a convex object with a boundary that is not $\mathcal{C}^1$-continuous, with (b) its convex subset (drawn in a thick curve) with a $\mathcal{C}^1$-continuous boundary, and (c) another subset with $\mathcal{C}^1$-continuous boundary that more closely approximates the original convex object.

Definition 3 *For a configuration $\boldsymbol{q}$ of $\mathcal{B}$, consider two pointed robots located at $\mathbf{p}_1$ and $\mathbf{p}_2$ such that $|\mathbf{p}_1\mathbf{p}_2| < W$ and $\mathcal{B}(\boldsymbol{q}) \cap G = \emptyset$, where $G = L - \overline{\mathbf{p}_1\mathbf{p}_2} + \{\mathbf{p}_1, \mathbf{p}_2\}$ denotes a straight wall with an opening gap from $\mathbf{p}_1$ to $\mathbf{p}_2$ and L denotes an infinite line supporting $\mathbf{p}_1$, $\mathbf{p}_2$ and the wall G. We define $\mathcal{H}^+(\mathbf{p}_1, \mathbf{p}_2, \boldsymbol{q})$ to be the half plane (separated by L) that the object $\mathcal{B}$ that is initially at the configuration $\boldsymbol{q}$ is confined and restricted by the wall from moving completely to the other half plane (Figure 5(a)).*

Capturing an object requires more restrictive constraint than that provided by a gaped wall mentioned earlier. Capturing constraint must prevent the object from moving to infinity in any direction. The following lemma gives a sufficient condition for a team of pointed robots to impose such constraint on the convex object $\mathcal{B}$.

Lemma 2 *Consider a cycle formation of $n \geq 3$ pointed robots surrounding the object $\mathcal{B}$ which is at a configuration $\boldsymbol{q}$. Let $\mathbf{p}_i, i = 1, 2, ..., n$ be the positions of the robots in the cycle in counterclockwise order and let* $\mathrm{next}(i) = \begin{cases} i+1 \text{ if } i < n, \\ 1 \; otherwise \end{cases}$ *denote the index of the robot successive to robot i. If $|\boldsymbol{p}_i\boldsymbol{p}_{\mathrm{next}(i)}| < W$ for $i = 1, 2, ..., n$ and $\mathcal{H} = \bigcap_{i=1,2,...,n} \mathcal{H}^+(\boldsymbol{p}_i, \boldsymbol{p}_{\mathrm{next}(i)}, \boldsymbol{q}) \neq \emptyset$, the object $\mathcal{B}$ cannot escape from the formation.*

PROOF: Assume oppositely that the object can escape. Because $\mathcal{H} \neq \emptyset$, by Definition 3, for the object $\mathcal{B}$ to escape to infinity, it has to pass through one of the n openings (each of which is formed by a pair of robots at $\boldsymbol{p}_i$ and $\boldsymbol{p}_{\mathrm{next}(i)}, i \in \{1, 2, ..., n\}$). Such successful passage contradicts Lemma 1 because $|\boldsymbol{p}_i\boldsymbol{p}_{\mathrm{next}(i)}| < W$. The lemma is therefore proved. ■

3 Capturing an Object

The discussion up to this point has been concerned only with convex objects. In the following lemma, we give a sufficient condition for capturing a nonconvex object. This lemma straightforwardly extends Lemma 2 using an intuitive fact that when part of a rigid object is captured, the whole object is captured as well.

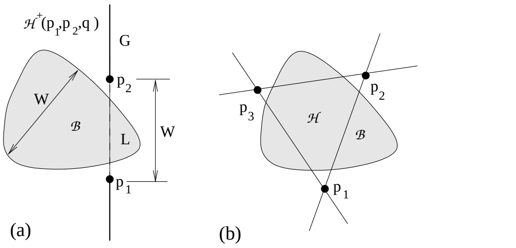

Figure 5: (a) the object is confined in the left half plane by a gapped wall, and (b) the object is captured in a cycle formation of the robots.

Lemma 3 *Let $\mathcal{A}$ be a nonconvex rigid object with a convex subset $\mathcal{B}$. Consider a cycle formation of pointed robots that do not intersect with the object $\mathcal{A}$. If the formation satisfies Lemma 2 with the convex subset $\mathcal{B}$, then the formation captures the object $\mathcal{A}$ (Figure 6).*

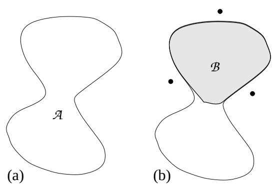

Figure 6: (a) a nonconvex object, (b) the object is captured when its convex subset (shaded region) is captured.

So far we have assumed dimensionless pointed robots. For two disc-shaped robots with the same nonnegative radius r, it is easy to see that the shortest distance between the robots when their centers are far apart by distance d is equal to $d - 2r$. Taking this into account, to handle disc-shaped robots with a common radius $r \geq 0$, we generalize Lemma 2 by replacing W with $W + 2r$. Accordingly, the involved condition has to be rewritten as $|\boldsymbol{p}_i \boldsymbol{p}_{\text{next}(i)}| < W + 2r$ where $\boldsymbol{p}_i$ denotes the position of the robot's center. This ensures that for every pair of consecutive robots in the cycle, there exists a pair of points (one for each robot) such that the distance between the two points is less than W (see Figure 7). As explained in the proof of Lemma 2, this condition forbids the object from escaping.

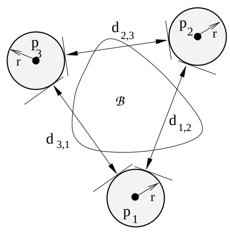

Figure 7: Capturing the object using three robots with a common radius r and the shortest distance between consecutive robots $d_{i,j} < W$.

At this point, we have accomplished the main objective of the paper. That is, we have presented a sufficient condition for a team of disc-shaped robots to capture an object in the plane. In the remainder of this section, we will sketch how a capturing formation could be computed. We plan to propose an efficient algorithm for this problem in our future paper.

We will assume pointed robots and limit our discussion to the case of an object whose boundary is explicitly given. Our approach is derived directly from Lemma 3. It consists of two steps: (1) find a convex subset of the given object and compute the width of the subset, and (2) compute the robots' positions satisfying Lemma 2 with the convex subset found in step one.

Finding a Convex Subset

First we find an inner polygonal approximation of the object. An inner polygonal approximation of a given figure is a polygon that is completely inside, and approximates the figure. The inner polygonal approximation can be found using a variation in two dimension of the polygonization technique in [4]. Clearly the resulting polygon is a subset of the given object.

Next, we compute a convex subset of the polygon. Any subset of this polygon is obviously a subset of the given object as well. To find a convex subset of the polygon, we resort to a computational geometry technique for convex partitioning [8]. Convex partitioning problem asks how a polygon can be partitioned into a small number of convex pieces (Figure 8(b)). An algorithm giving the smallest number of partitions is presented in [2]. Using this heuristic makes sense because having fewer partitions usually implies larger partitions with larger width value. For each partition obtained, its width can be computed by applying the algorithm for computing the diameter function of a polygon in [3].

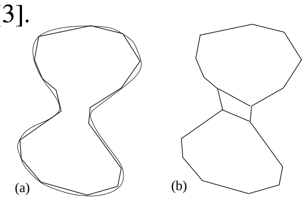

Figure 8: (a) inner polygonal approximation of the curve object shown in Figure 6(a), and (b) its convex partition.

Computing the Robots' Positions

With a given convex subset of the object and its width, we can follow Lemma 3 to generate positions of the robots that can capture the object. What arrangement of robots is preferred depends on target application of the resulting capturing formation. Here we will show few examples.

An obvious arrangement is to place the robots on a circle enclosing the given object while ensuring that the distance between every pair of consecutive robots is smaller than the width of the convex subset (Figure 9(b)). A smaller enclosing circle is preferred because it results in fewer robots needed. A simple way to find an enclosing circle is to compute a polygon bounding the object (Figure 9(a)) and use the algorithm in [12] to compute the smallest circle enclosing the bounding polygon. Many robots may be needed to cover the entire enclosing circle but selecting the position of each robot is trivial because the circle does not intersect with the object.

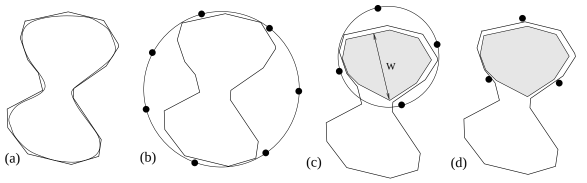

Figure 9: (a) a polygon bounding the object, (b)-(d) various arrangement of robots in a capturing formation (see text).

Another arrangement is to place the robots on a circle enclosing the convex subset. The positions must avoid intersecting with the object and again the distance between every pair of consecutive robots must be smaller than the width of the convex subset (Figure 9(c)). Figure 9(d) shows another arrangement in which the robots are placed along the boundary of the bounding polygon that encloses the convex subset. This type of arrangement requires fewer robots than the others while allowing less freedom of the object's motion in the capture.

4 Manipulating an Object

In this section, we sketch a simple approach that applies capturing formation to the problem of object manipulation. This is only to show a sample application of capturing formation. We are currently working on developing an efficient algorithm for this problem with implementation using mobile robots and plan to present our result in the near future.

We will explain the notion of independent capturing discs and sketch how a manipulation plan can be generated using this workspace-based capturing concept. Due to space limitation, we discuss only the case of pointed robots transporting a convex polygonal object (can be extended to handle more general cases by applying the ideas presented in previous sections).

4.1 Independent Capturing Discs

Let us consider a convex polygon $\mathcal{B}$ with the width W. Assume that we are given a capturing formation of the polygon using n pointed robots, each of which is placed on a circle enclosing the polygon at position $\boldsymbol{p}_i, i \in \{1, 2, .., n\}$ (Figure 10(a)). Let us construct a cycle of n discs with a common radius r, namely $\Omega_i, i = 1, 2, .., n$, that surround the polygon by having disc Ω_i touch the circle at $\boldsymbol{p}_i$. It is easy to see that when two discs of a common radius r have their centers far apart by distance less than $W - 2r$, the distance between any pair of points (one from each disc) must be less than W (capturing condition). Based on this condition, we can compute, e.g., using a numerical method, the largest value of the common radius r at which the distance between any pair of points from every pair of consecutive discs in the cycle is less than W (Figure 10). We therefore obtain n discs such that a capturing formation can be generated by placing each robot anywhere in each disc. We refer to these discs as **I**ndependent **C**apturing **D**iscs or in short as ICDs and a capturing formation generated as such is said to belong to the corresponding ICDs.

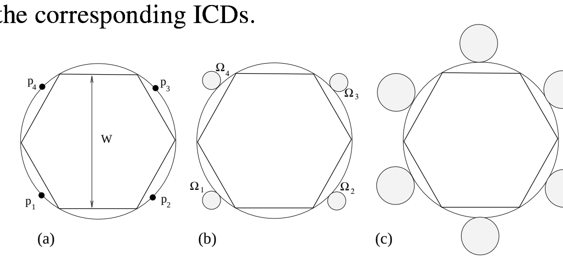

Figure 10: (a) a formation of 4 robots capturing a hexagon, (b) ICDs (drawn as shaded discs) constructed from this formation, and (c) larger ICDs with two additional robots.

In contrast with the concept of MICaDs in [10], our ICDs are not limited for three robots. By adding extra robots and appropriately arranging the corresponding ICDs, it is easy to see that we generally can increase the size of ICDs allowing greater freedom of the capturing formation (Figure 10(c)). This is due to the pairwise nature of the distance constraint on consecutive robots in the formation.

4.2 Moving Cage

Each robot can move simultaneously and independently in each of the ICDs while ensuring that the enclosed object cannot escape. This property allows us to directly apply the motion planning in an obstacle-free workspace given in [10] by replacing their MICaDs with our ICDs. In the rest of this section, we summarize the main idea of this technique. In the following sketch, we say "a capturing formation for a configuration $\boldsymbol{q}$" to refer to a capturing formation that can capture the object that is at the configuration $\boldsymbol{q}$. We also say "ICDs for a configuration $\boldsymbol{q}$" to refer to a set of ICDs of which all capturing formations can capture the object that is at the configuration $\boldsymbol{q}$.

Overlapping ICDs

For n robots and a configuration $\boldsymbol{q}$ of $\mathcal{B}$, clearly if n is sufficiently large, we can compute a set of ICDs $\Omega = \{\Omega_i\}, i = 1, 2, ..., n$ associated with this configuration (e.g., using the method presented in the previous section). It is obvious that a set of ICDs for another configuration can be immediately obtained by applying to ICDs Ω the rigid transformation that maps $\boldsymbol{q}$ to this configuration (because the distance between any pair of points is preserved under a rigid transformation therefore the capturing ability remains intact). Now let us consider a set of ICDs $\Omega' = \{\Omega'_i\}, i = 1, 2, .., n$ for a configuration $\boldsymbol{q}'$ that is obtained by applying to Ω the rigid transformation that maps $\boldsymbol{q}$ to $\boldsymbol{q}'$. Assume further that $\Omega_i \cap \Omega'_i \neq \emptyset, i = 1, 2, ..., n$ (we can always find $\boldsymbol{q}'$ in the neighborhood of $\boldsymbol{q}$ such that this is true, see [10] for a proof of this statement). Now consider the following plan: (1) Move robot i to anywhere in disc Ω_i $(i = 1, 2, ..., n)$, and (2) Move robot i along any trajectory within disc Ω_i to anywhere in the region $\Omega_i \cap \Omega'_i, (i = 1, 2, ..., n)$. In step one the robots move to a capturing formation for the configuration $\boldsymbol{q}$. By confining the motion of the robots in ICDs Ω, the object cannot escape during the entire motion in step two and when this step is completed, the robots will form a capturing formation for the configuration $\boldsymbol{q}'$.

Manipulation Plan

By connectedness of the object configuration space, we can compute a sequence of overlapping ICDs starting from ICDs for an initial configuration and ending at ICDs for a target configuration. With this sequence, step two in the above plan can be repeatedly executed to move robots from an overlap to the next one in the sequence. The robots will reach a capturing formation for the target configuration and because the robots are commanded to be in ICDs at all times, the object is prevented from escaping during the entire manipulation as if it is transported in a moving cage.

Limitation

The manipulation plan described above does not require the robots to maintain a fixed formation, i.e., they have some freedom to move with respect to one another in the ICDs. This allows manipulation to be accomplished without precise synchronization of the robots. However, as mentioned in [10], the success of the manipulation assumes that jamming does not occur. Also note that this manipulation does not guarantee to bring the object exactly to the target configuration. It takes the object from the initial configuration to a configuration in a neighborhood of the target configuration. This manipulation is therefore useful when precisely reaching a final configuration is not a critical issue, e.g., when an object need to be moved from an initial configuration to any configuration in a specified region of the workspace.

5 Conclusion

We have presented in this paper a sufficient condition for a team of disc-shaped robots to capture an object in the plane. We have also proposed to use capturing formation as a basis for object manipulation.

A convex partitioning technique from computational geometry is used for computing a convex subset in Section 3. This approach does not guarantee that a convex subset with the largest width will always be found. We plan to present a more effective approach in a future paper. We are also interested in finding a method for computing maximal ICDs and a strategy to produce a jam-free manipulation plan.

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