Introduction to Quantum Computing

by
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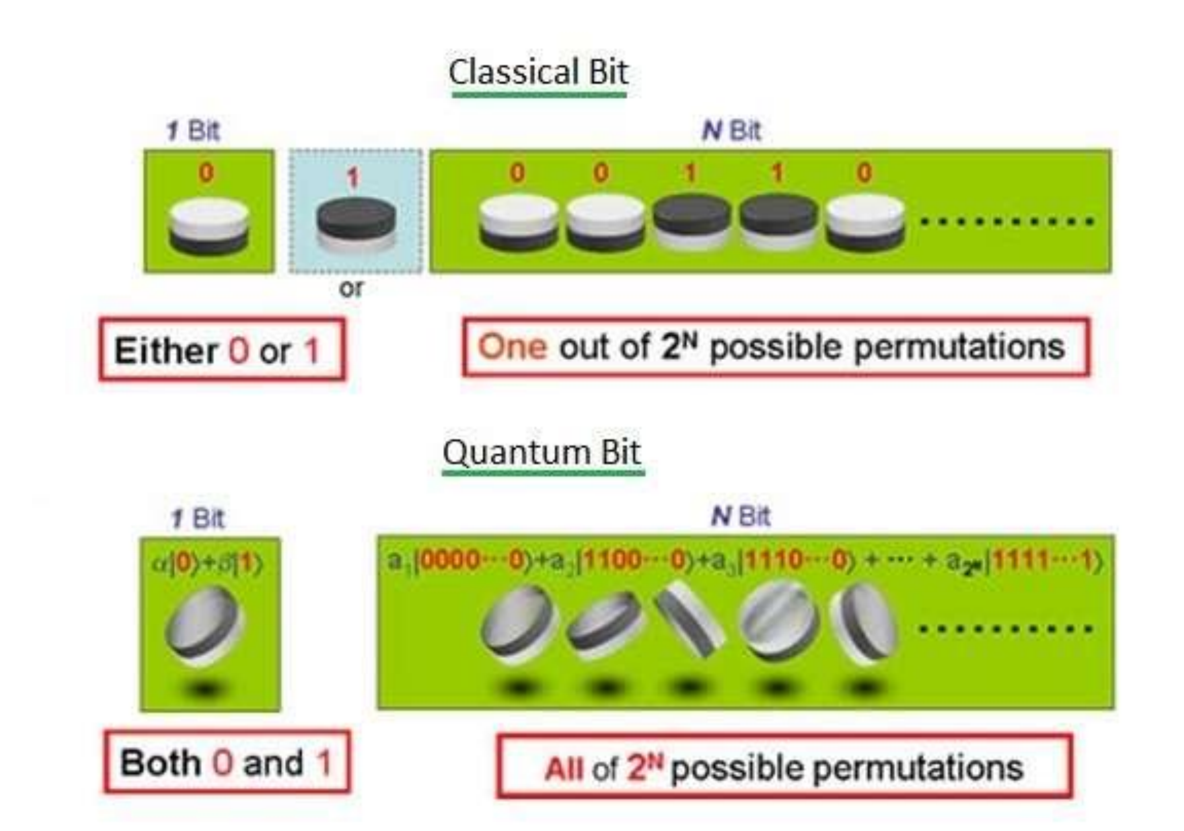
Overview

- From bits to qubits: Dirac notation, density matrices, measurement, Bloch sphere
- Quantum circuits: basic single-qubit & two-qubit gates, multipartite quantum states
- Entanglement: Bell states, Teleportation, Superdense coding
- Quantum algorithms: Deutsch-Jozsa algorithm, Grover's algorithm

From bits to qubits



- Superpositions allow to perform calculations on many states at the same time.
  - ➢ Quantum algorithms with **exponential speed-up**.
- But: Once we measure the superposition state, it collapse to one of its states.
- We can use **interference effects** to keep the right answer.

Photo courtesy of : https://medium.com/qntm/qntm-entering-the-era-of-quantum-computing

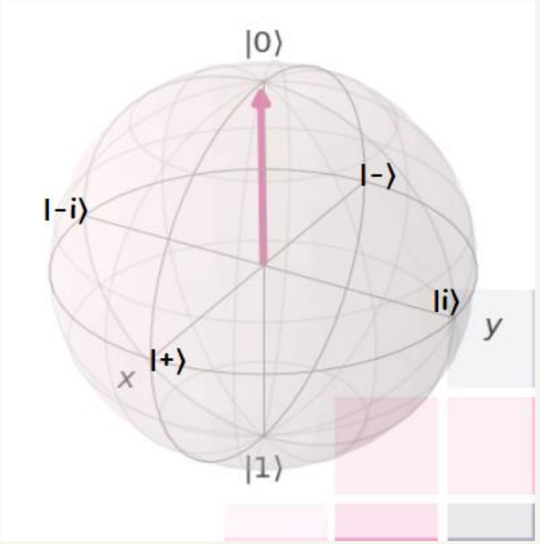
Dirac notation & density matrices

- It used to describe quantum states: Let a, b are 2-dimensional vector with complex entries.
  - ket: $|a\rangle = \binom{a_0}{a_1}$
  - bra: $\langle b| = |b\rangle^+ = \binom{b_0}{b_1}^+ = (b_0^* \; b_1^*)$
  - bra-ket: $\langle b|a\rangle = a_0 {b_0}^* + a_1 {b_1}^* = \langle a|b\rangle^* \in \mathbb{C}$ (inner product)
  - ket-bra: $|a\rangle\langle b| = \begin{pmatrix} a_0 {b_0}^* & a_0 {b_1}^* \\ a_1 {b_0}^* & a_1 {b_1}^* \end{pmatrix}$ (2x2 matrix)

Dirac notation & density matrices

- The pure states are $|0\rangle = \binom{1}{0}, |1\rangle = \binom{0}{1}$, which are orthogonal: $\langle 0|1\rangle = 1.0 + 0.1 = 0$
- $|0\rangle\langle 0| = \binom{1}{0}(1\ 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, |1\rangle\langle 1| = \binom{0}{1}(0\ 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
- $\mathrm{P} = \begin{pmatrix} \mathrm{P}_{00} & \mathrm{P}_{01} \\ \mathrm{P}_{10} & \mathrm{P}_{11} \end{pmatrix} = \mathrm{P}_{00}|0\rangle\langle 0| + \mathrm{P}_{01}|0\rangle\langle 1| + \mathrm{P}_{10}|1\rangle\langle 0| + \mathrm{P}_{11}|1\rangle\langle 1|$
- All quantum states can be described by density matrices.
- All quantum states are normalized, i.e., $\langle\psi|\psi\rangle = 1$, e.g., $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$
- A density matrix is pure, if $\mathrm{P} = |\psi\rangle\langle\psi|$, otherwise it is mixed.
  - $\mathrm{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 0| \rightarrow \mathrm{Pure}, \mathrm{P} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = |1\rangle\langle 1| \rightarrow \mathrm{Pure}$
  - $\mathrm{P} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \rightarrow \mathrm{Mixed}$
  - $\mathrm{P} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}(|0\rangle - |1\rangle)(|0\rangle - |1\rangle) \rightarrow \mathrm{Pure}$

Measurement



- We choose orthogonal base to describe and measure quantum states (projective measurement).
- During a measurement onto the basis $\{|0\rangle, |1\rangle\}$, the states will collapse into either state $|0\rangle$ or $|1\rangle$, as those are the eigenstates of σ_Z, we call this a Z-measurement.
- Other different bases are:
  - $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, corresponding to the eigenstates of σ_x,
  - $|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$, $|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$, corresponding to the eigenstates of σ_y.

Measurement

- **Born rule**: the probability that a state $|\psi\rangle$ collapses during a project measurement onto the basis $\left\{|X\rangle,\ \left|X^{\perp}\right\rangle\right\}$ to the state $|X\rangle$ is given by $P(X) = |\langle X|\psi\rangle|^2$, $\sum_i P\,(X_i) = 1$
- Examples:
  - ➢ $|\psi\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + \sqrt{2}|1\rangle\right)$ is measured in the basis $\{|0\rangle, |1\rangle\}$:

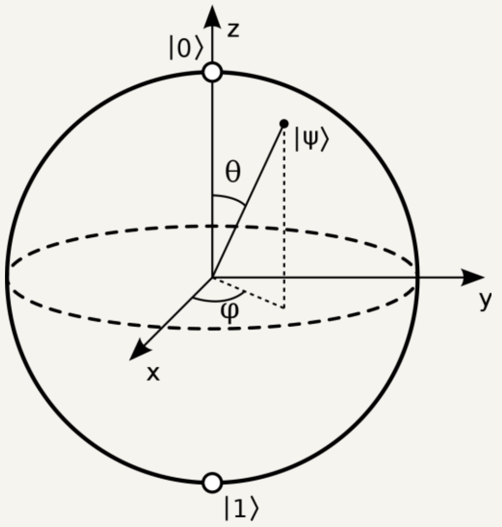
$$P(0) = \left\langle 0\right|\frac{1}{\sqrt{3}}(|0\rangle + \sqrt{2}\left|1\right\rangle)^2 = \left|\frac{1}{\sqrt{3}}\langle 0|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}\langle 0|1\rangle\right|^2 = \frac{1}{3} \ \rightarrow \ P(1) = \frac{2}{3}$$

  - ➢ $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ is measured in the basis $\{|+\rangle, |-\rangle\}$:

$$P(+) = |\langle +|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right|^2 = \frac{1}{4}|(\langle 0|0\rangle - \langle 0|1\rangle + \langle 1|0\rangle - \langle 1|1\rangle)|^2 = 0 \rightarrow \text{expected as } \langle +|-\rangle = 0,$$

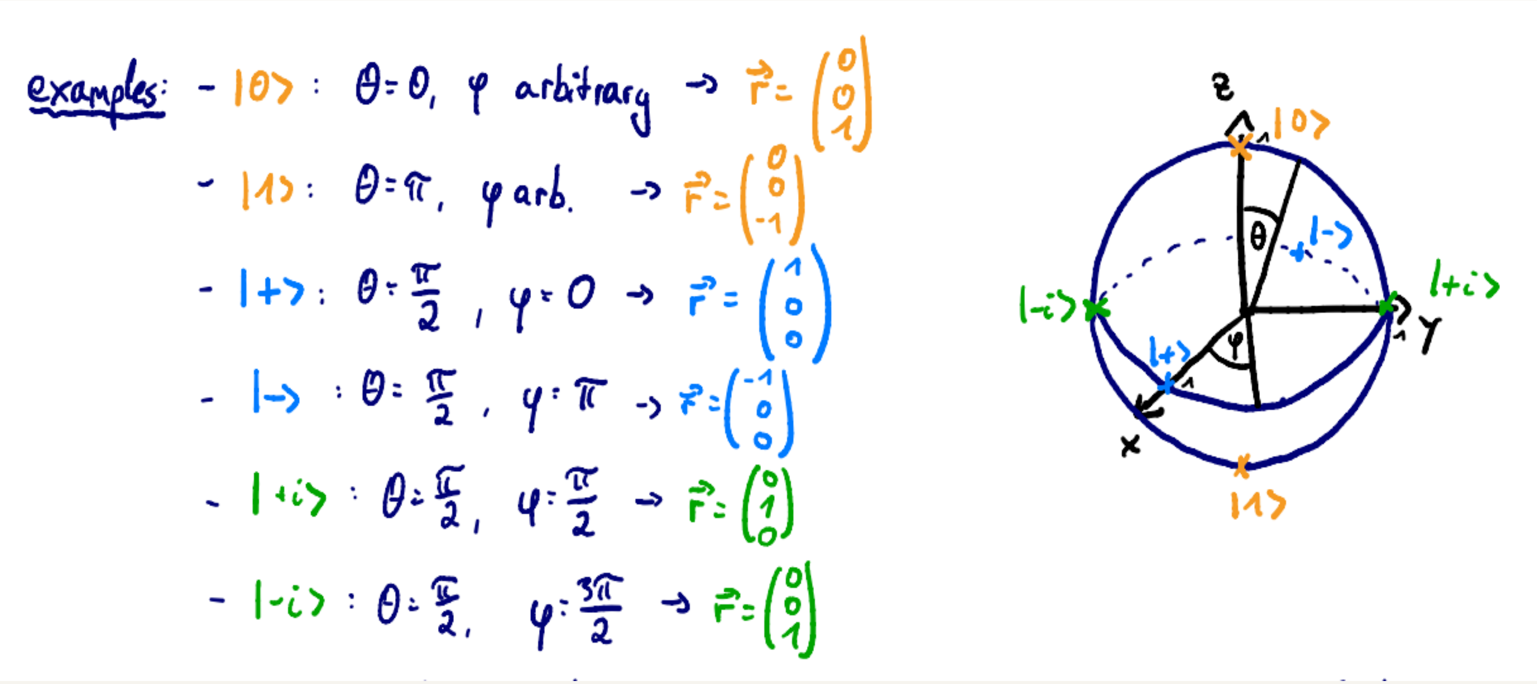
$$P(-) = |\langle -|-\rangle|^2 = 1$$

Bloch sphere



- We can write any normalized **pure state** as $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$, where $\varphi \in [0, 2\pi]$ describes the relative phase and $\theta \in [0, \pi]$ determines the probability to measure $|0\rangle, |1\rangle$: $P(|0\rangle) = \cos^2\frac{\theta}{2}$, $P(|1\rangle) = \sin^2\frac{\theta}{2}$.
- All normalized pure states can be illustrated on the surface of a sphere with radius $|\vec{r}| = 1$, which we call the **Bloch sphere**.
- The coordinates of such a state are given by the Bloch vector: $\vec{r} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$

Bloch sphere



- **Be careful**: On the Bloch sphere, angles are twice as big as in Hilbert space:
  - ➢ e.g., $|0\rangle$ & $|1\rangle$ are orthogonal, but on the Bloch sphere their angle is 180^o.
  - ➢ For a general state, $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \cdots \rightarrow \theta$ is the angle on the Bloch sphere, while $\frac{\theta}{2}$ is the actual angle in Hilbert space!

Photo courtesy of : IBM quantum summer school 2019

Quantum circuits: single qubit gates

- **Circuit model**: sequence of building block that carry out computations, called **gates**.

input — algorithm — output

- **Quantum gates** are represented by unitary matrices, A unitary matrix is a square matrix of complex numbers, whose inverse is equal to its conjugate transpose.
- **Single qubit gates**:

	Hadamard	H	$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$	← creates superposition
rotation around X-axis by π →	Pauli-X	X	$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$	← bit flip
rotation around Y-axis by π →	Pauli-Y	Y	$\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$	← bit & phase flip
rotation around Z-axis by π →	Pauli-Z	Z	$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$	← phase flip
	Phase	S	$\begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}$	← used to change from Z to Y-basis
	$\pi/8$	T	$\begin{bmatrix}1 & 0\\ 0 & e^{i\pi/4}\end{bmatrix}$	

Quantum circuits: single qubit gates

- $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|$

  ↳ $\sigma_x|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$, $\sigma_x|1\rangle = (|0\rangle\langle 1| + |1\rangle\langle 0|) \cdot |1\rangle = |0\rangle\underbrace{\langle 1|1\rangle}_{1} + |1\rangle\underbrace{\langle 0|1\rangle}_{0} = |0\rangle$

- $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$

  ↳ $\sigma_z|+\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = |-\rangle$, $\sigma_z|-\rangle = (|0\rangle\langle 0| - |1\rangle\langle 1|) \cdot \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$

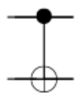
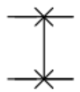
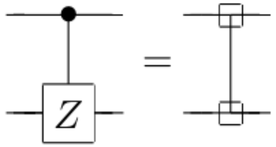
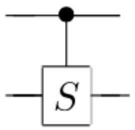
- Hadamard gate: one of the most important gates for quantum circuits

  $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} (|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$

  ↳ $H|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = |+\rangle$, $H|1\rangle = \frac{1}{\sqrt{2}} (|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|) \cdot |1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$

Photo courtesy of : IBM quantum summer school 2019

Quantum circuits: multiple-qubit gates

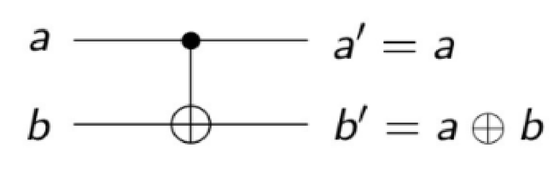
controlled-NOT		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
swap		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
controlled-Z		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$
controlled-phase		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{bmatrix}$
Toffoli		$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

Quantum circuits: two-qubit gates

- Classical example: XOR

irreversible: given the output, we cannot recover the input.

- But as quantum theory is unitary, we only consider unitary and therefore reversible gates
- Quantum example: CNOT gate



a	b	a'	b'
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

Quantum circuits can perform all function that can be calculated classically.

Quantum circuits: multipartite quantum states

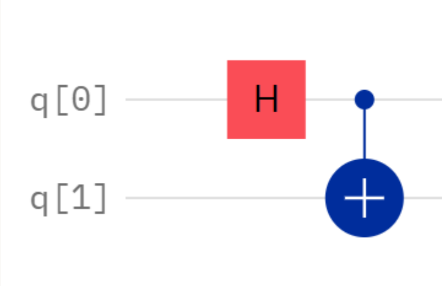
- We use tensor product to describe multiple states:
  - ➢ $|a\rangle \otimes |b\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix}$
  - ➢ Example: system A is in state $|1\rangle_{\mathrm{A}}$ and system B is in state $|0\rangle_{\mathrm{B}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, states of this form are called **uncorrelated**.
  - ➢ But there are also bipartite states that cannot be written as $|\psi\rangle_a \otimes |\psi\rangle_b$. These states are **correlated** and sometimes even entangled (very strong correlation), e.g. $|\psi\rangle_{\mathrm{AB}}^{(00)} = \frac{1}{\sqrt{2}}(|00\rangle_{\mathrm{AB}} + |11\rangle_{\mathrm{AB}}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, it so called Bell state, used for teleportation, cryptography, Bell tests, etc.

Entanglement

- If a pure state$|\psi\rangle_{\mathrm{AB}}$on system A,B cannot be written as $|\psi\rangle_a \otimes |\phi\rangle_b$, it is entangled.
- These are four so called **Bell states** that are maximally entangled and build on orthonormal basis:
  - $\left|\psi^{\mathbf{00}}\right\rangle := \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(|00\rangle + |11\rangle)$,
  - $\left|\psi^{\mathbf{01}}\right\rangle := \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(|01\rangle + |10\rangle)$,
  - $\left|\psi^{\mathbf{10}}\right\rangle := \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(|00\rangle - |11\rangle)$,
  - $\left|\psi^{\mathbf{11}}\right\rangle := \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(|01\rangle - |10\rangle)$.

Entanglement

- Creation of Bell states:



$$|q_0q_1\rangle_{00}\ H_0 \rightarrow \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(|00\rangle + |10\rangle)\ CNOT_{\mathbf{01}} \rightarrow \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(|00\rangle + |11\rangle) = |\psi^{\mathbf{00}}\rangle,$$

$$|q_0q_1\rangle_{01}\ H_0 \rightarrow \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(|01\rangle + |11\rangle)\ CNOT_{\mathbf{01}} \rightarrow \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(|01\rangle + |10\rangle) = |\psi^{\mathbf{01}}\rangle,$$

$$|q_0q_1\rangle_{10}\ H_0 \rightarrow \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(|00\rangle - |10\rangle)\ CNOT_{\mathbf{01}} \rightarrow \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(|00\rangle - |11\rangle) = |\psi^{\mathbf{10}}\rangle,$$

$$|q_0q_1\rangle_{11}\ H_0 \rightarrow \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(|01\rangle - |11\rangle)\ CNOT_{\mathbf{01}} \rightarrow \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(|01\rangle - |10\rangle) = |\psi^{\mathbf{11}}\rangle.$$

Teleportation

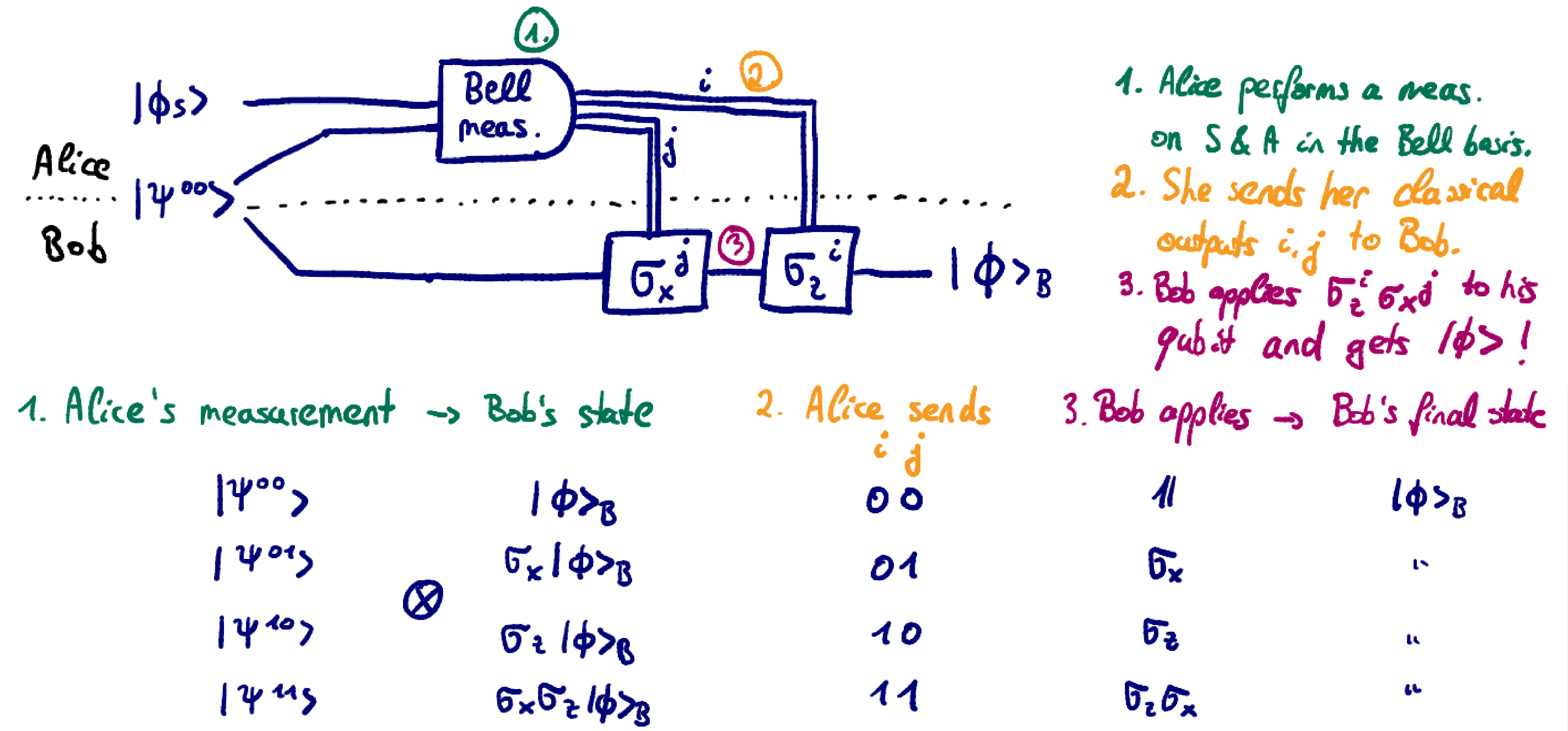
- Goal:
  - Alice want to send her (unknown) state $|\phi\rangle_s := \alpha|0\rangle_s + \beta|1\rangle_s$ to Bob.
  - She can only send him two classical bits though.
  - They both share the maximally entangled state $|\psi\rangle_{AB}^{(00)} = \frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB})$.
- Initial states of the total system:

$$
\begin{aligned}
|\phi\rangle_S \otimes |\Psi^{00}\rangle_{AB} &= \frac{1}{\sqrt{2}}\left(\alpha|000\rangle_{SAB} + \alpha|011\rangle_{SAB} + \beta|100\rangle_{SAB} + \beta|111\rangle_{SAB}\right) \\
&= \frac{1}{2\sqrt{2}}\Big[\left(|00\rangle_{SA} + |11\rangle_{SA}\right)\otimes\left(\alpha|0\rangle_B + \beta|1\rangle_B\right) + \left(|01\rangle_{SA} + |10\rangle_{SA}\right)\otimes\left(\alpha|1\rangle_B + \beta|0\rangle_B\right) \\
&\quad + \left(|00\rangle_{SA} - |11\rangle_{SA}\right)\otimes\left(\alpha|0\rangle_B - \beta|1\rangle_B\right) + \left(|01\rangle_{SA} - |10\rangle_{SA}\right)\otimes\left(\alpha|1\rangle_B - \beta|0\rangle_B\right)\Big] \\
&= \frac{1}{2}\Big[|\Psi^{00}\rangle_{SA}\otimes|\phi\rangle_B + |\Psi^{01}\rangle_{SA}\otimes\left(\sigma_x|\phi\rangle_B\right) \\
&\quad + |\Psi^{10}\rangle_{SA}\otimes\left(\sigma_z|\phi\rangle_B\right) + |\Psi^{11}\rangle_{SA}\otimes\left(\sigma_x\sigma_z|\phi\rangle_B\right)\Big]
\end{aligned}
$$

Photo courtesy of : IBM quantum summer school 2019

Teleportation

- Protocol:

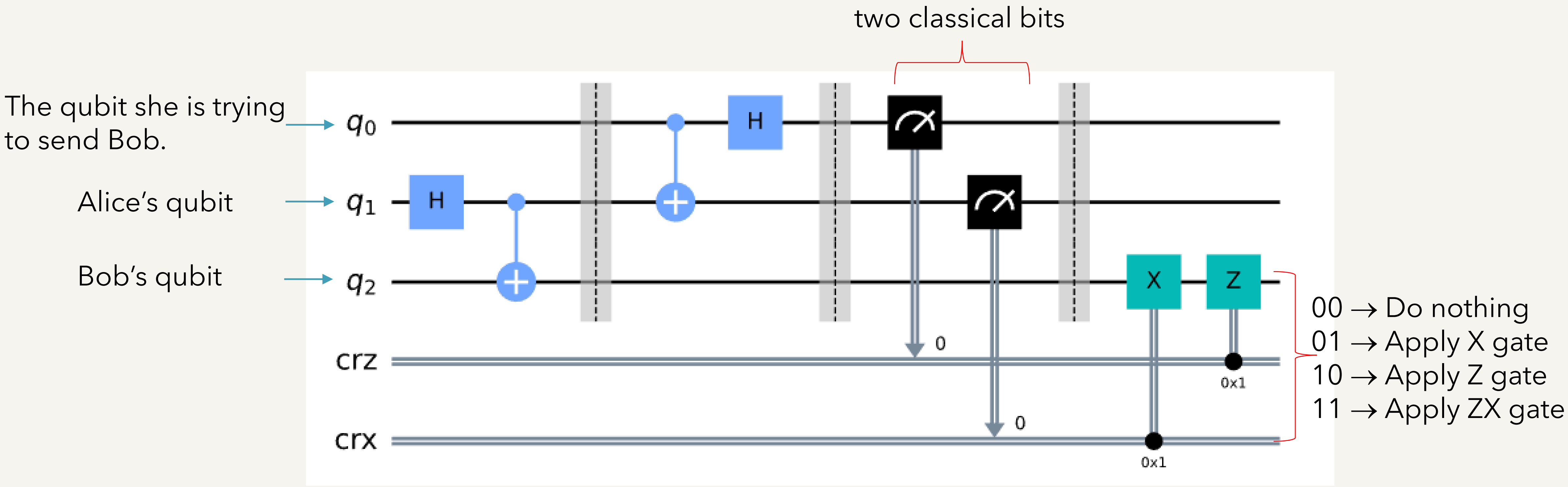


- Alice's state collapsed during the measurement, so she doesn't have the initial state $|\phi\rangle_S$ anymore.
  This is expected due to the no-cloning theorem, as she cannot copy her state, but just send her state to Bob when destroying her own.

Photo courtesy of : IBM quantum summer school 2019

Teleportation

- Quantum circuit:

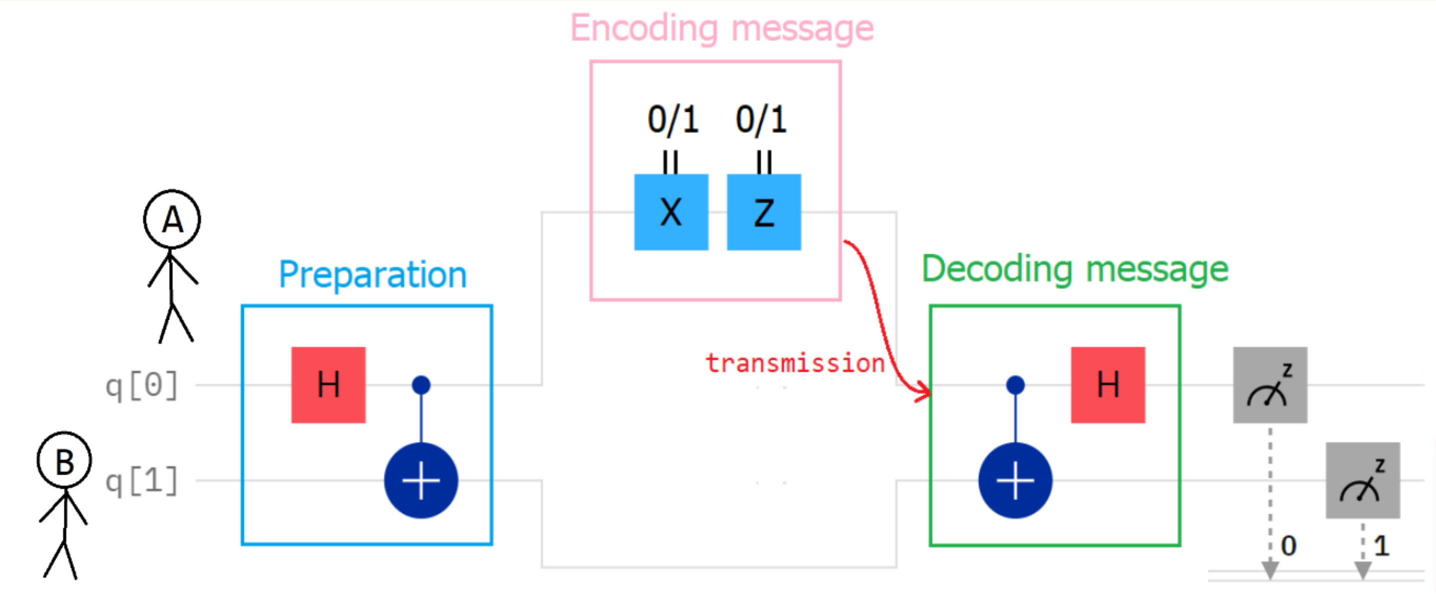


Superdense coding

- Superdense coding is a procedure that allows someone to send two classical bits to another party using just a single qubit of communication.
- Take advantage of quantum mechanics to more efficiently transmit classical information.
- Word "coding" means there are 2 essential processes, encoding and decoding:
  - ➢ encoding: classical state → quantum state,
  - ➢ decoding: quantum state → classical state.

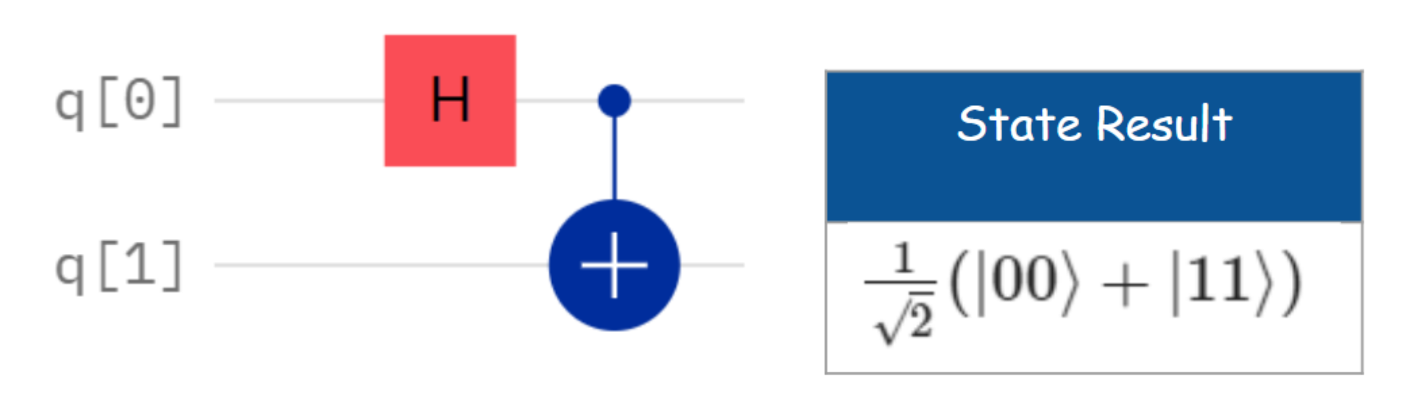
Teleportation	Superdense Coding
Transmit **one qubit** using **two classical bits**.	Transmit **two classical bits** using **one qubit**.

Superdense coding



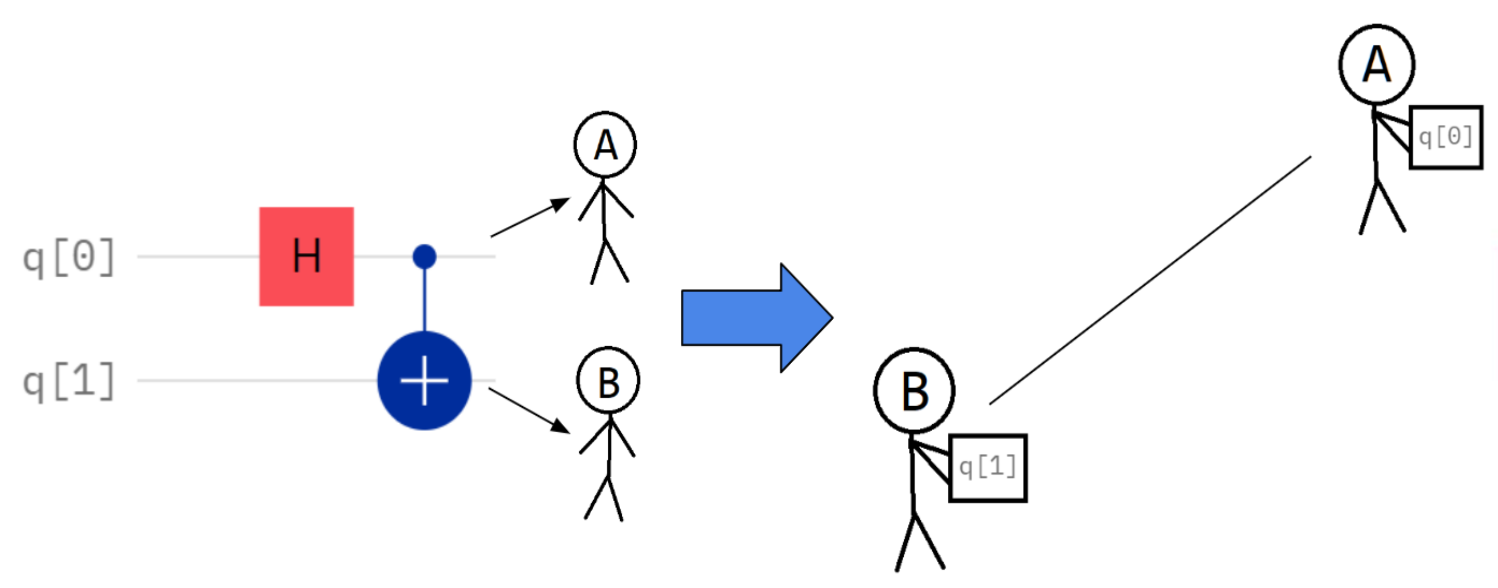
- Superdense coding includes 4 steps:
  - preparation,
  - encoding message,
  - transmission,
  - decoding message.

Superdense coding



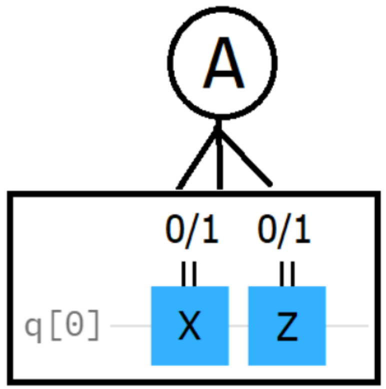
- Step 1: preparation
  - ➢ Start with 2 qubits in the basis state $|0\rangle$.
  - ➢ Applying Hadamard gate to the first qubit and CNOT gate (the first qubit as control, another qubit as target) accordingly.

Superdense coding



- Step 1: preparation
  - Give the first qubit to A and the second qubit to B.
  - A and B travel far away.

Superdense coding



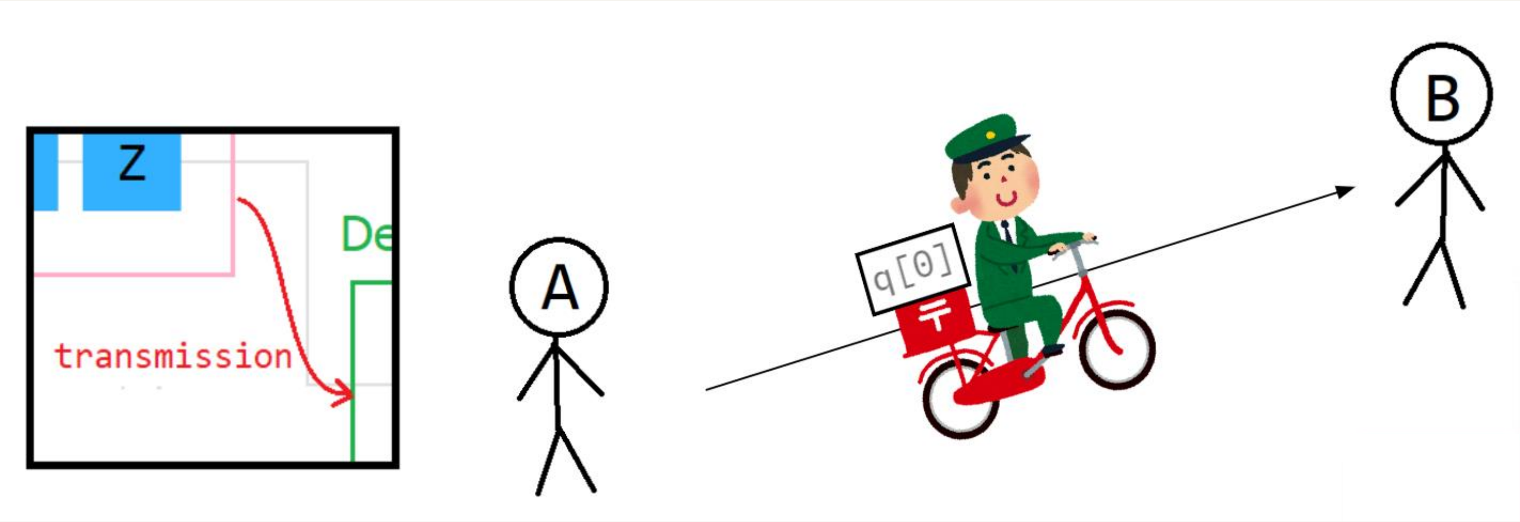
Message	Applied Gate	State Result
00	I	$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$
01	X	$\frac{1}{\sqrt{2}}(\|10\rangle + \|01\rangle)$
10	Z	$\frac{1}{\sqrt{2}}(\|00\rangle - \|11\rangle)$
11	ZX	$\frac{1}{\sqrt{2}}(\|10\rangle - \|01\rangle)$

Message	Applied Gate
00	
01	X
10	Z
11	X Z

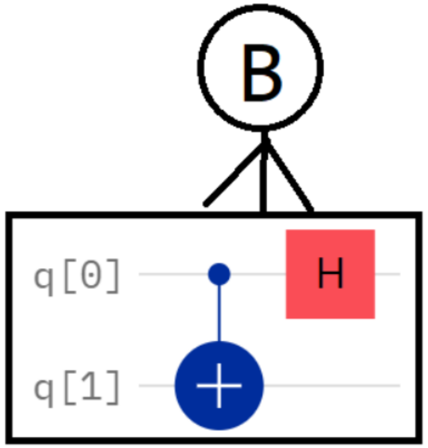
- Step 2: encoding message
  - ➢ A encodes the classical state in the qubit by applying gate(s).

Superdense coding

- Step 3: transmission
  - ➢ A sends the first qubit to B.



Superdense coding

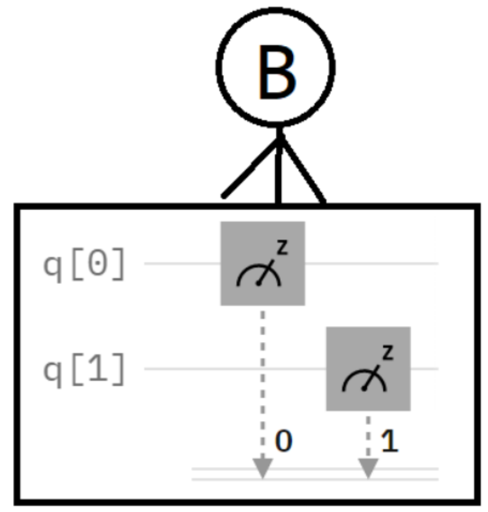


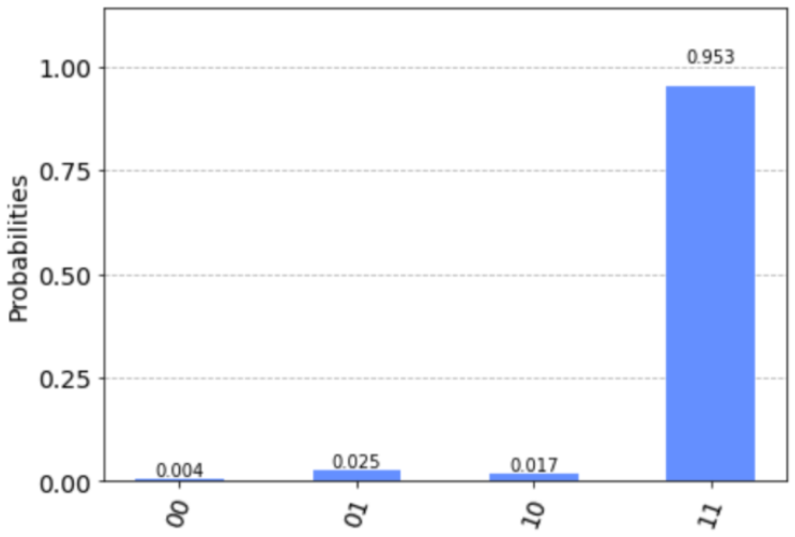
Message	State Result
00	$\|00\rangle$
01	$\|01\rangle$
10	$\|10\rangle$
11	$\|11\rangle$

- Step 4: decoding message
  - ➢ Applying CNOT gate (the first qubit as control, another qubit as target) and Hadamard gate to the first qubit accordingly.

Superdense coding

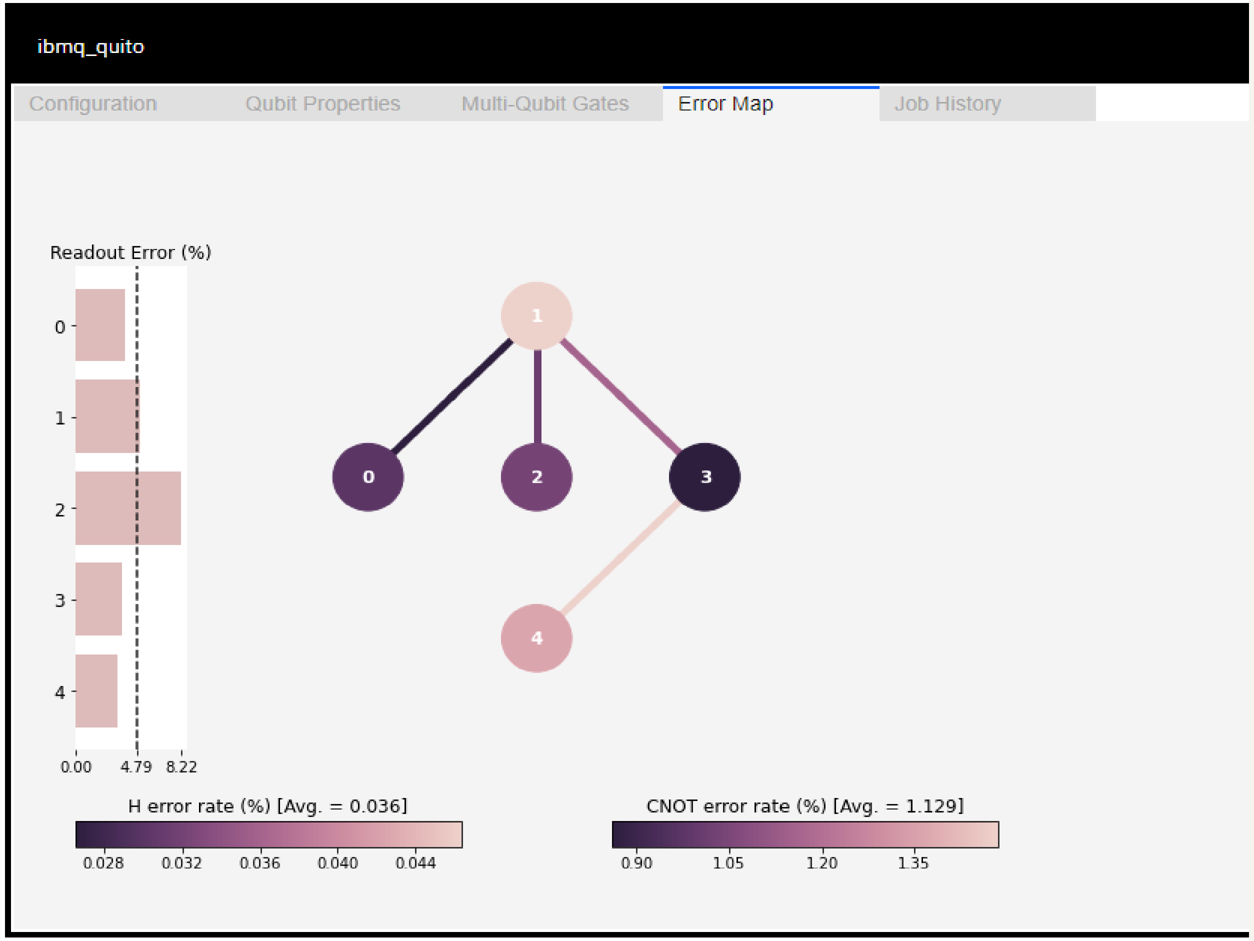
Test the circuit which encodes message "**11**" and run on "**ibm_oslo**".





- Step 4: decoding message
  - Finally, measure both qubits.

How the noise properties affect the result



- There are often optimizations that the transpiler can perform that reduce the overall gate count, and thus total length of the input circuits.
- Qiskit library has a command "backend" to show the chosen backend information graphically such as "Error Map".
- We can select a good initial layout considering connectivity and error information that you can find from the map to initial layout onto the physical qubits with at least noise.

Assignment I: Basic Quantum Computing

- Required:
  - Go to https://quantum-computing.ibm.com/
  - Register IBMid account or sign in with Google, Github, LinkedIn, or Twitter.
  - Download source codes at Assignment and upload files "**Lab-1.ipynb**", "**Lab-2.ipynb**" and "**Lab-3.ipynb**" into IBM Quantum Lab.
- Assignments:
  - Lab-1: Operations on single qubit and multiple qubits gates by IBM Quantum.
  - Lab-2: Quantum circuits by IBM Quantum.
  - Lab-3: Superdense coding.