จุฬาลงกรณ์มหาวิทยาลัย
CHULALONGKORN UNIVERSITY

Solving Optimization Problems With A Quantum Computer

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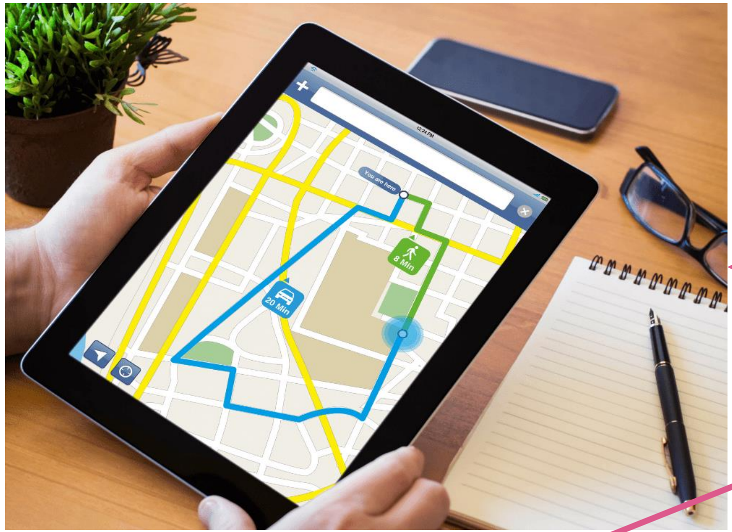
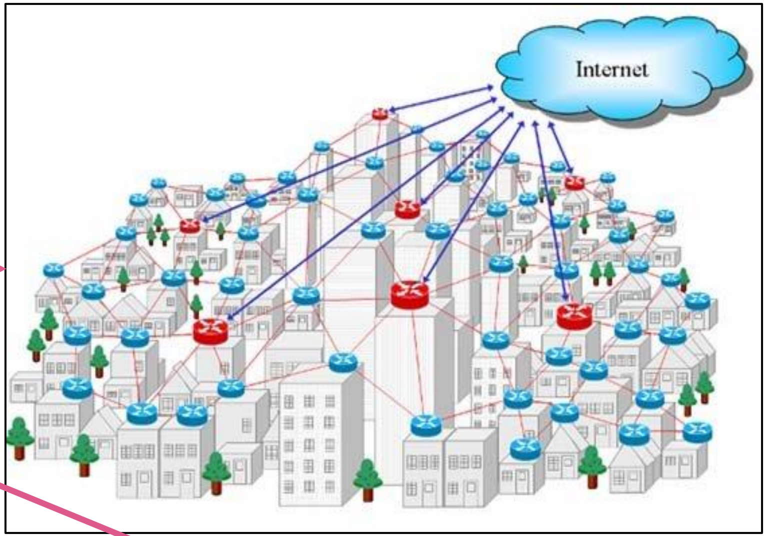
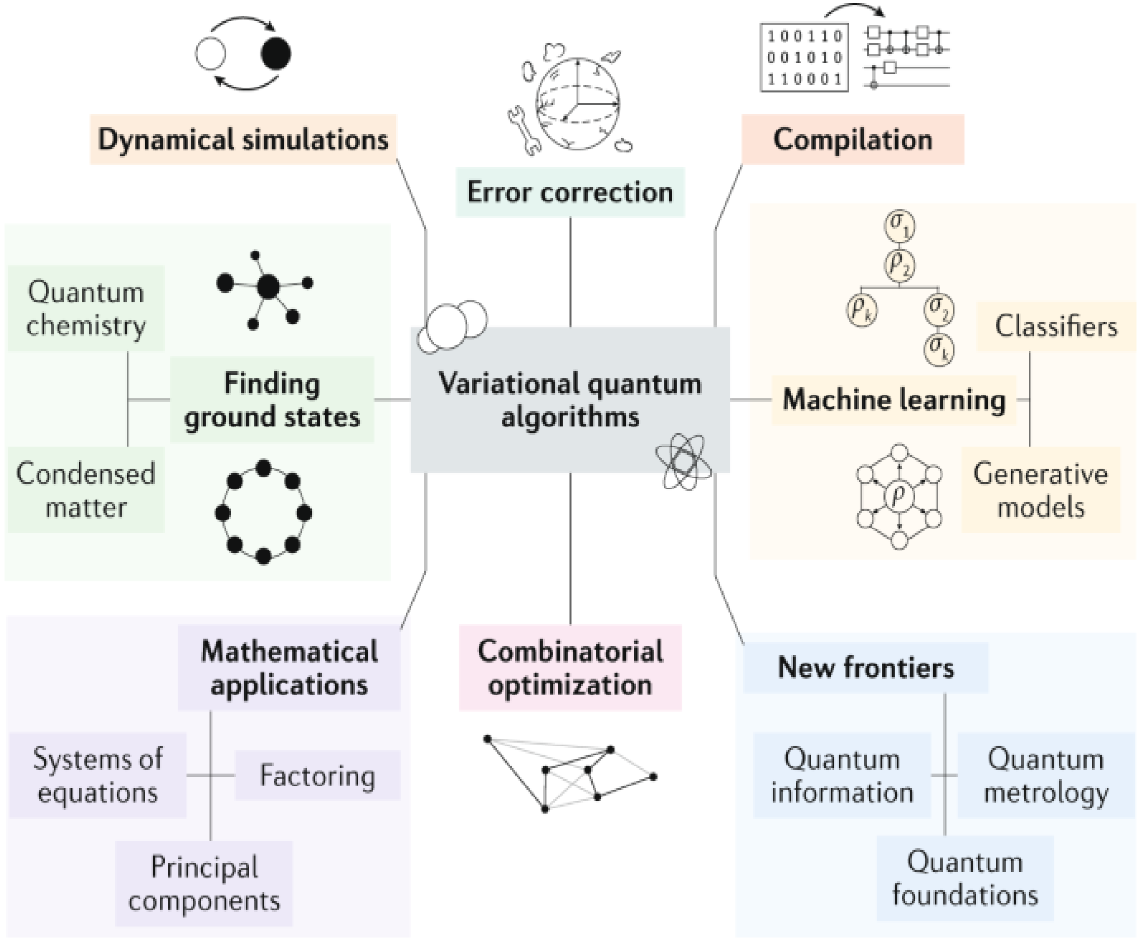
Toward solving optimization problems with a quantum computer

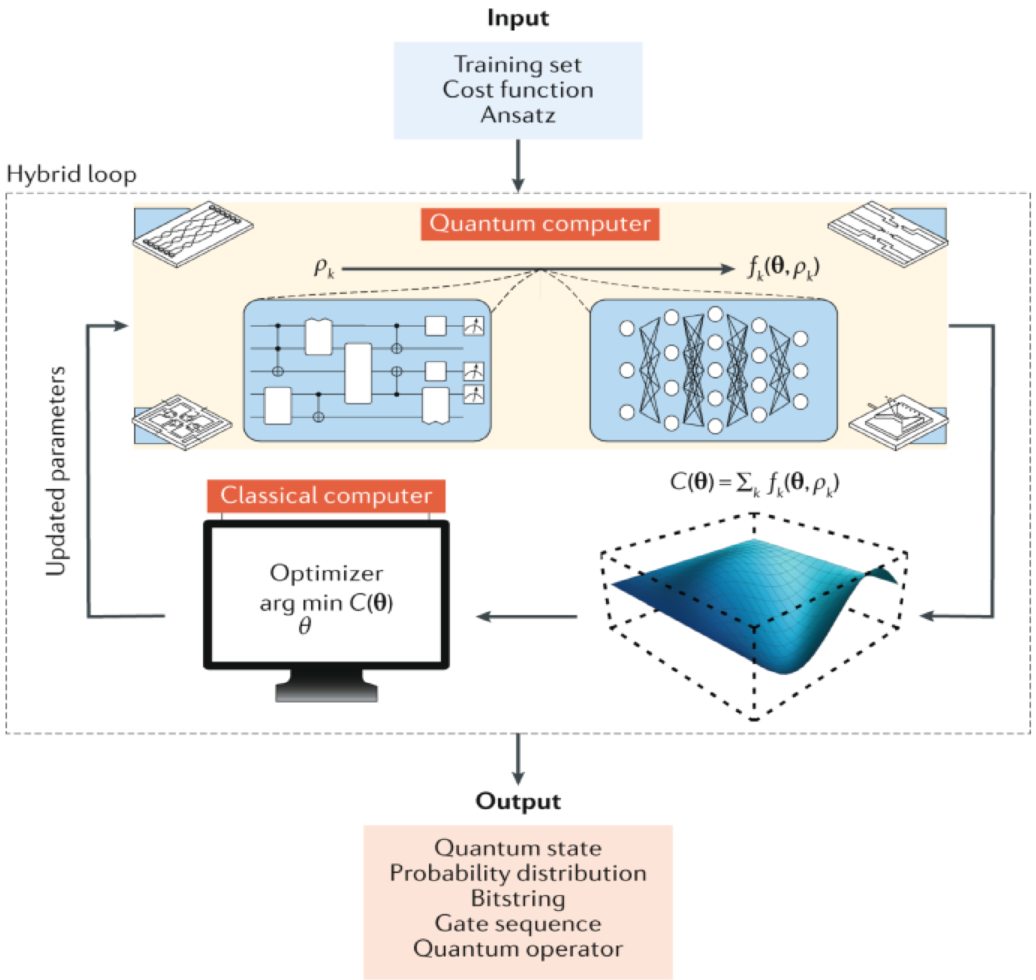


Optimization problems



Variational quantum algorithms





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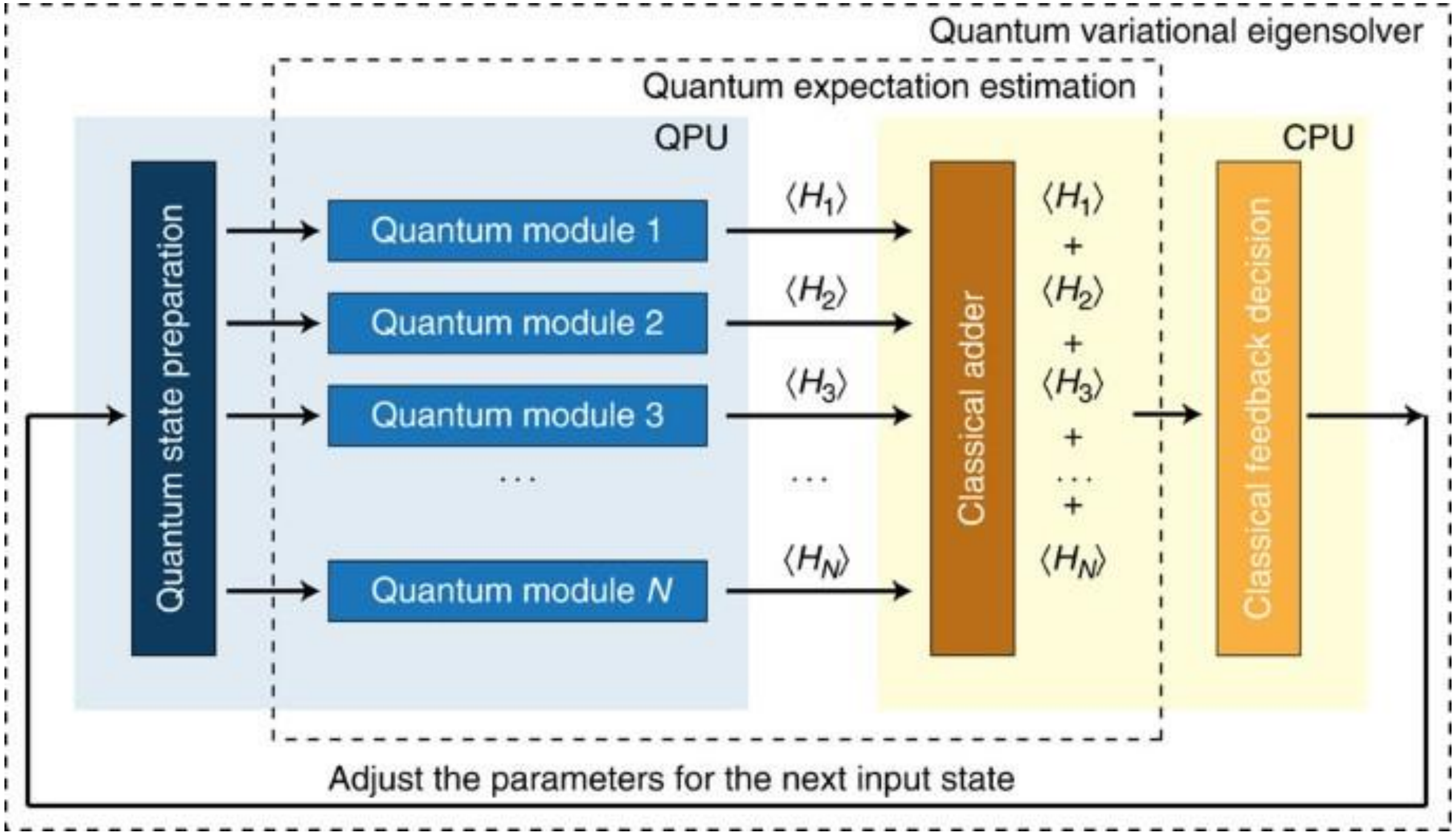
Quantum variational eigensolver



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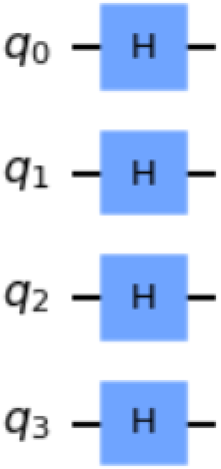
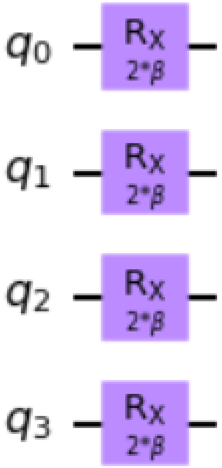
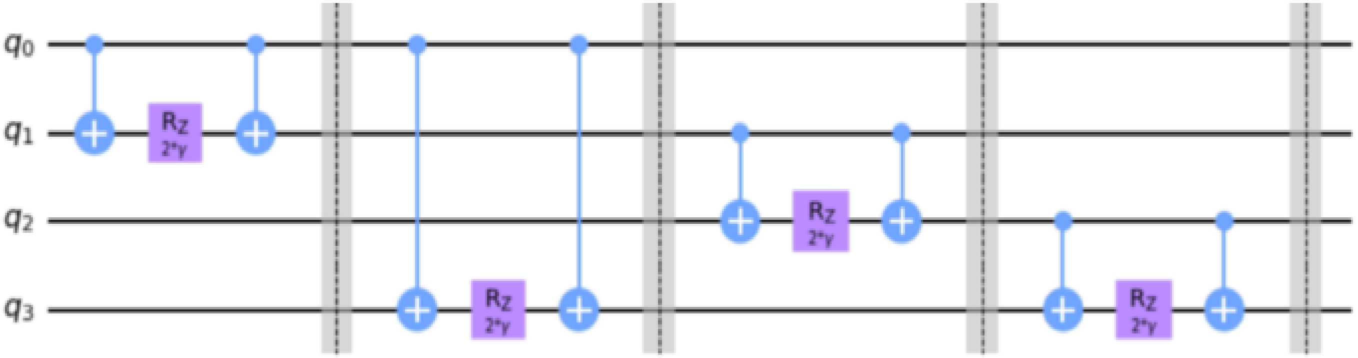
Quantum approximate optimization algorithm (QAOA)



Preparing an initial state



The mixing unitary



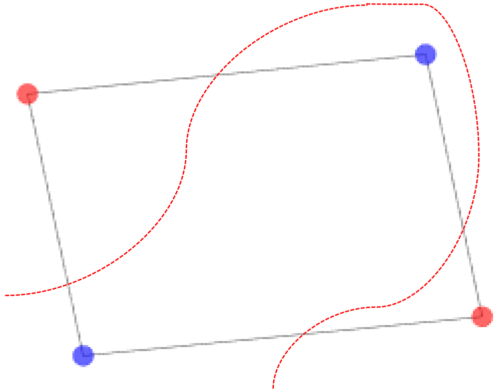
The problem unitary

$$|\psi(\beta,\gamma)\rangle = \underbrace{U(\beta)U(\gamma)\cdots U(\beta)U(\gamma)}_{p\text{ times}}|\psi_0\rangle$$

$U(\beta) = e^{-i\beta H_B}$ and $U(\gamma) = e^{-i\gamma H_P}$

where H_B is the mixing Hamiltonian and H_P is the problem Hamiltonian.

Goal: find $(\beta_{opt},\gamma_{opt})$ to get $\langle\psi(\beta_{opt},\gamma_{opt})|H_P|\psi(\beta_{opt},\gamma_{opt})\rangle$



The max-cut problem

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Quantum approximate optimization algorithm (QAOA)

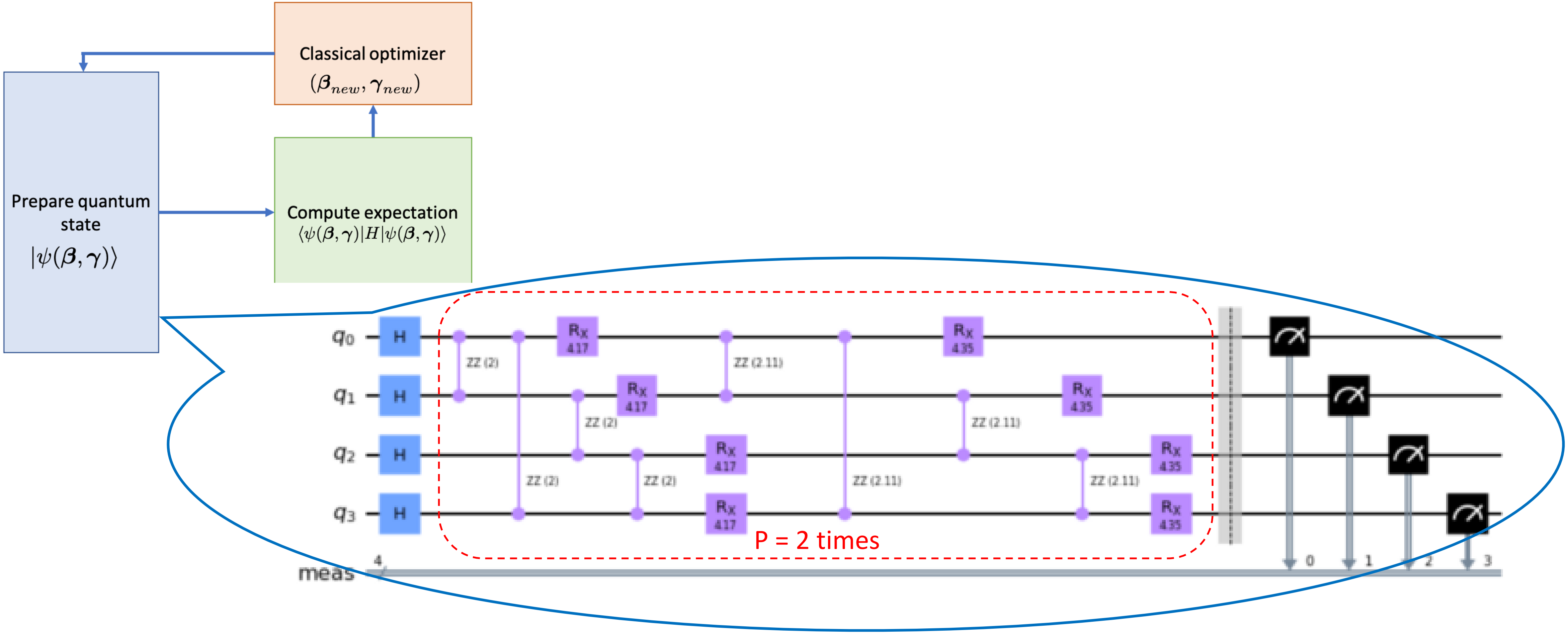


Photo courtesy of https://qiskit.org/textbook/ch-applications/qaoa.html

Quantum-inspired evolutionary algorithm

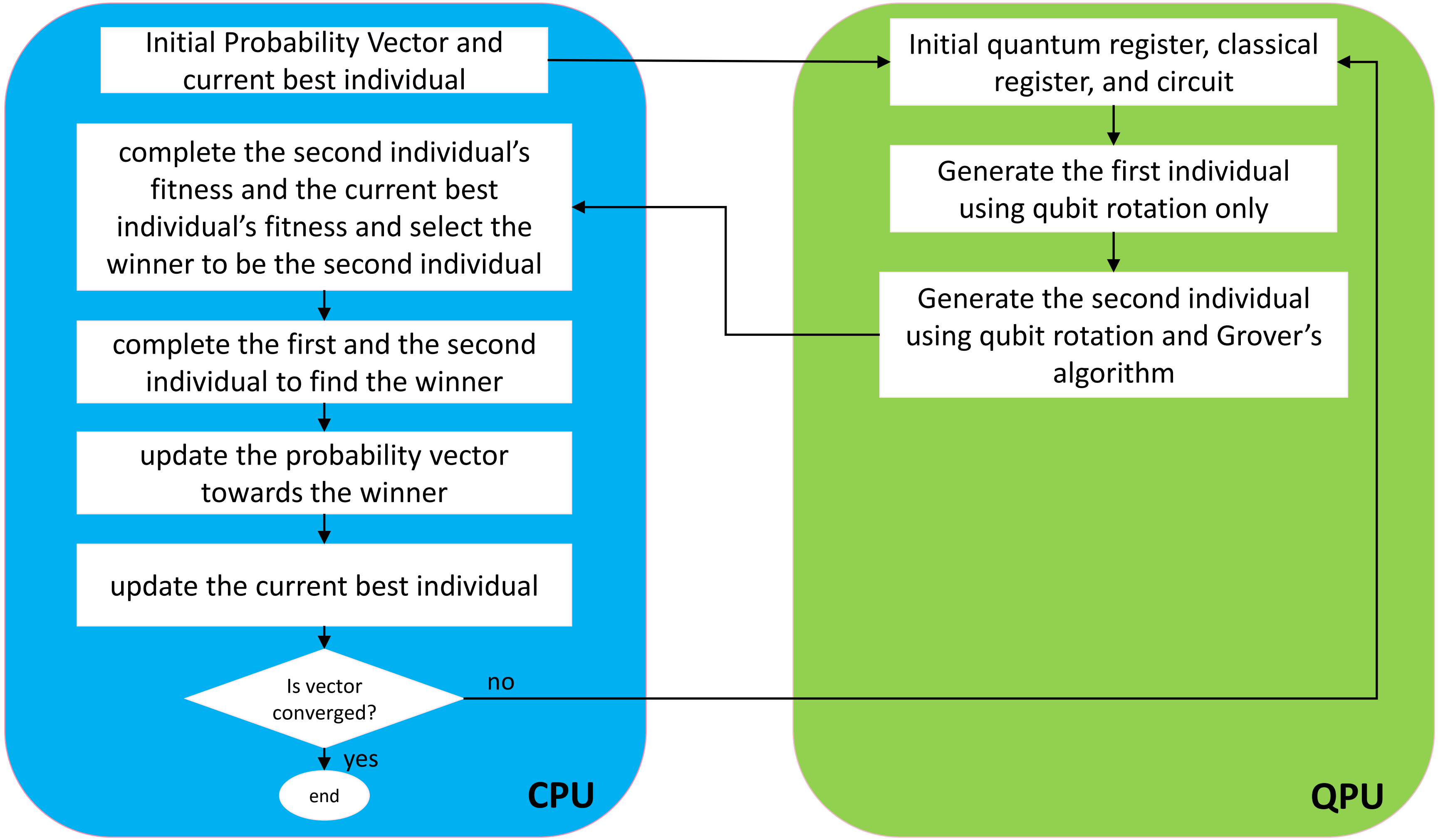
- The optimization problem is one of the most important problem in computer science and has a wide range of applications to engineering and finances. One approach to tackle the optimization problem is to use evolutionary algorithms such as Genetic algorithms (GAs) and Compact genetic algorithm (cGA).

- Quantum-inspired genetic algorithms are one of the attempt where the algorithms are still executed purely in the classical computers but took some of the ideas of quantum phenomena and translate them into classical analogue.

Quantum-assisted genetic algorithms

- Quantum-assisted genetic algorithms perform mutation operator while still performing crossover and population update on the classical side.
- Compact genetic algorithm represents the population as a a probability distribution in a quantum register. The quantum variable is then updated toward the winner via qubit rotation
- Redefining the GA in the context of quantum computation by creating a population with superposition of all states.

Grover-assisted Compact Genetic Algorithm (cGA*)



The schematic of Grover-assisted compact genetic algorithm.

Grover’s Search Algorithm

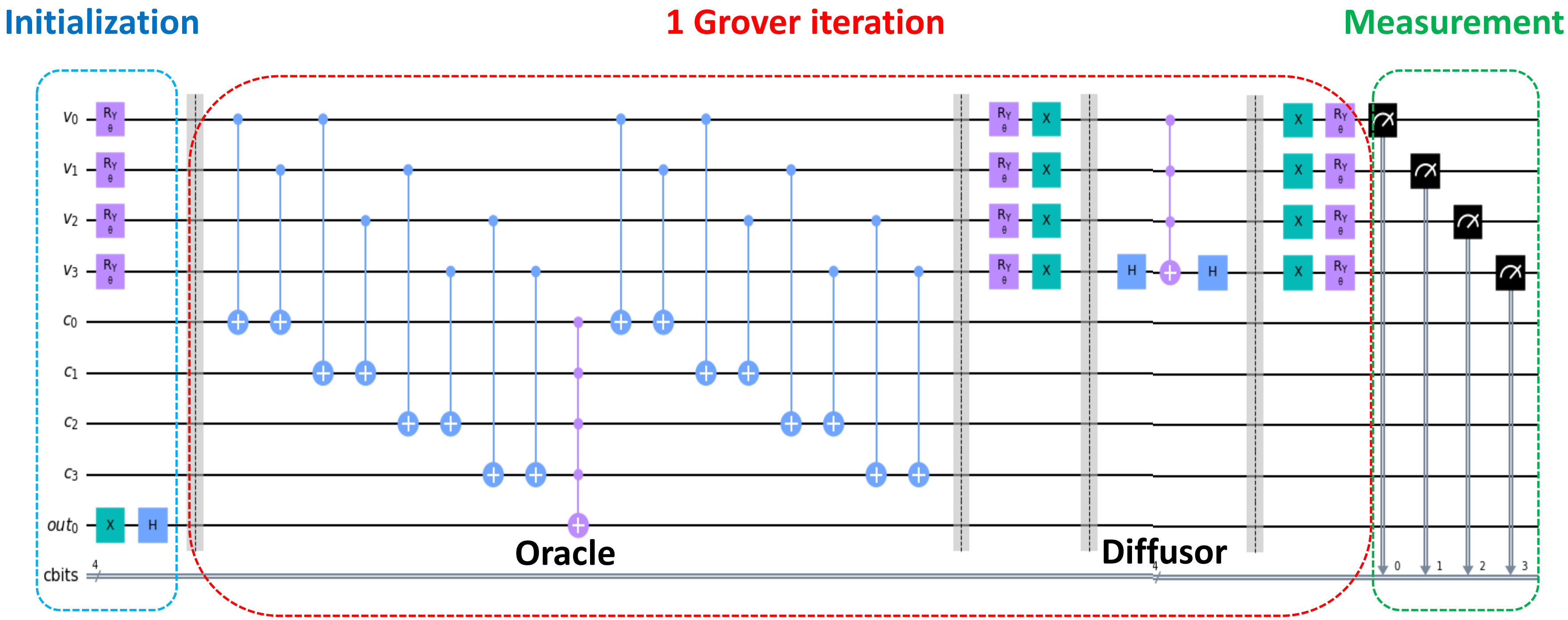
- “For what value of x does $f(x) = k$, for some number k?”
- A quantum computer allows us to calculate $f(x)$ for a superposition of all values of x at the same time, but not to extract useful information from that superposition directly.
- Instead, we use the result to increase the amplitude of the good values of x and repeat until we have a high probability of reading out the value we want.
- If there are N possible values for x, we will have to try half of them ($N/2$) on average before finding the answer.
- Grover's algorithm uses entanglement and interference to grow those probability amplitudes faster than a simple linear check of all possible values. Instead of $N/2$ tries, around $\sqrt{N}$ calculations will be enough.

The Stages of Grover's Search Algorithm

1. Initialization : All qubits are set to be in superposition. All states have the amplitude $1/\sqrt{N}$
2. Oracle : The oracle function marks the state *x′* that satisfies the condition *f(x′)* = 1 by performing a phase flip.
3. Amplification : The amplification stage phase flips the amplitudes around the average amplitude. As the target state's amplitude was inverted while the other states kept their original amplitudes, the flip causes the target state's amplitude to increase and the others to decrease.
4. Measurement : The qubits are read, and output given.

Grover iteration (repeat oracle and amplification stages) requires approximately $\frac{\pi}{4}\sqrt{\frac{N}{t}}$ where *N* is the number of states and *t* is the number of target solutions.

Grover-assisted Compact Genetic Algorithm (cGA*)



The quantum circuit of Grover's search algorithm.

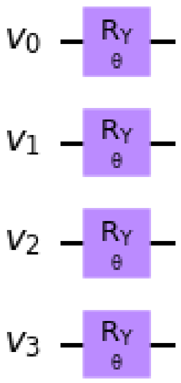
Grover-assisted Compact Genetic Algorithm (cGA*)

An arbitrary single-qubit state is defined:

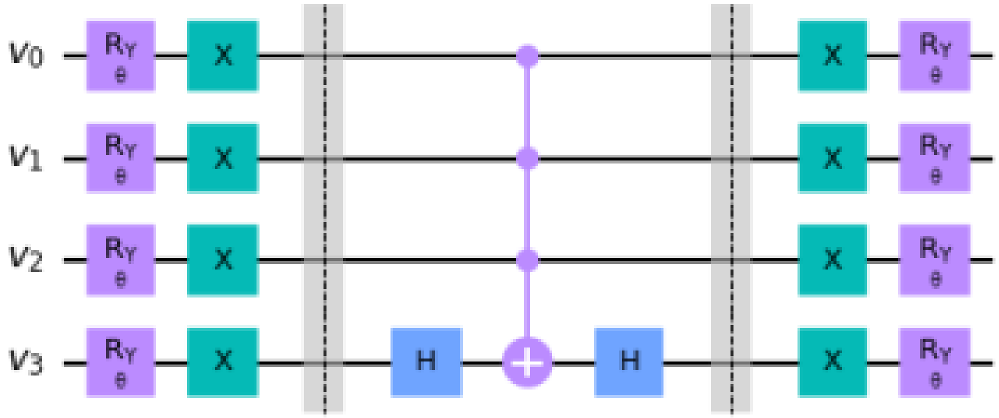
$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

The qubit rotation for angle(θ) is defined:

$$angle(\theta) = (probability(p) - 0.5) \times \pi$$



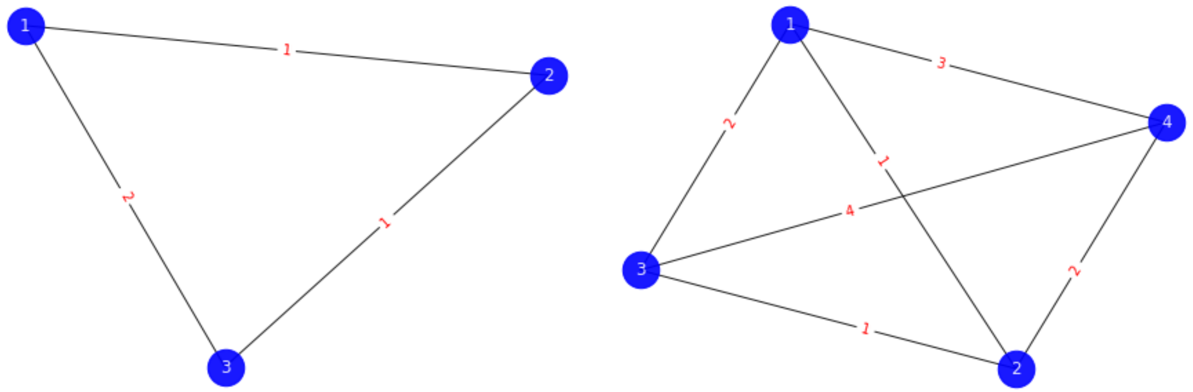
A circuit to initial the state based on the probability which encoding problem with 4 qubits.



A diffuser circuit which encoding problem with 4 qubits.

Travelling Salesman Problem

- The objective function is to find a shortest route that a salesman visits every city exactly once and returns to the starting point.
- In our oracle, we only focus on defining an oracle to recognize all feasible solutions.
- TSP is mapped to an Ising model with scales $(N-1)^2$ spins are required, where N is the number of cities, and we designate city 1 to appear first in the route.



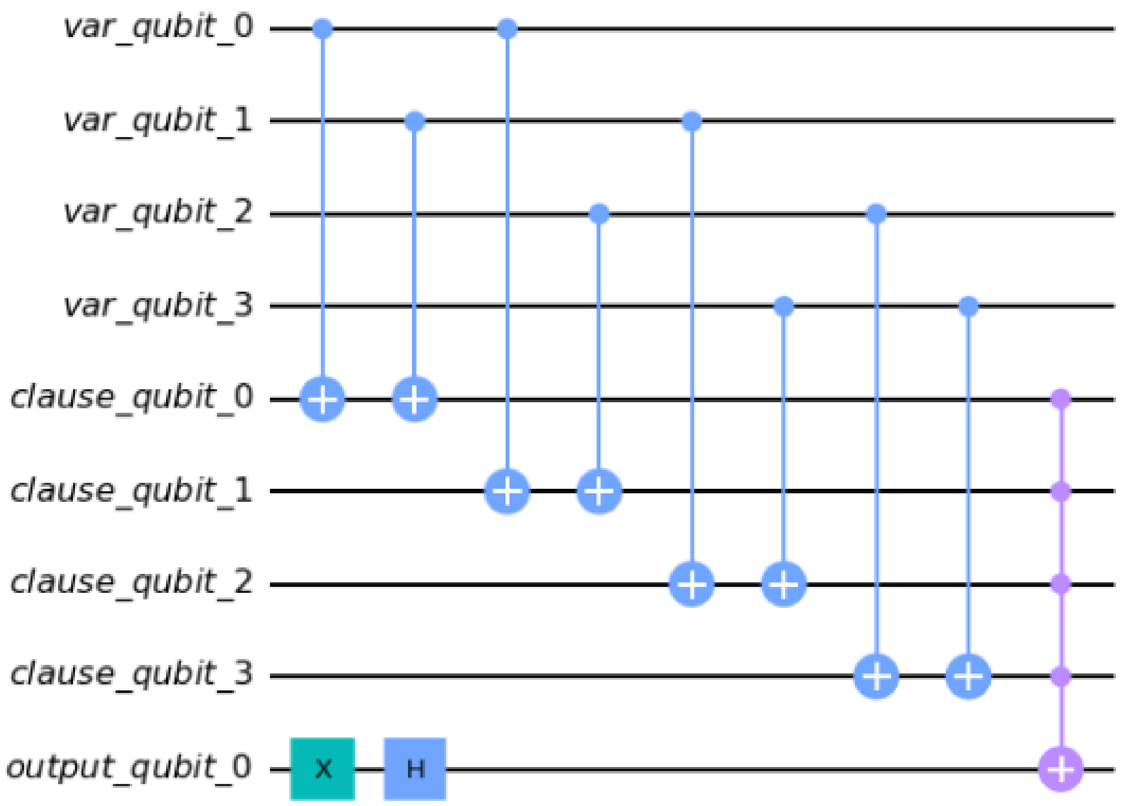
The TSP 3 cities and 4 cities.

	1	2	3
1	1	0	0
2	0	1	0
3	0	0	1

Matrix represents the route 1→ 2 → 3 → 4 → 1.

Travelling Salesman Problem

- The oracle to check a feasible solution on the quantum state:



An example of oracle circuit for TSP 3 cities.

Ex., the oracle to verify a solution of TSP 3-city, we need to check all 4 clauses include: $(X_{2,2}, X_{2,3})$, $(X_{2,2}, X_{3,2})$, $(X_{2,3}, X_{3,3})$, and $(X_{3,2}, X_{3,3})$.

Travelling Salesman Problem

- Classical optimizer:
  - In this paper, it will be presumed that the graph is fully connected.
  - As a result, our energy will have three components in a single objective function to be minimized, and we get the following:

$$\text{Energy} = \boxed{\sum_{i,j} w_{i,j} \sum_{p} X_{i,p} X_{j,p+1}} + \boxed{B \sum_{p} \left(1 - \sum_{i} X_{i,p}\right)^2 + B \sum_{i} \left(1 - \sum_{p} X_{i,p}\right)^2}$$

Total distance

The constraints to check a feasible path

where *B* is the weight of the penalty term, it has a large enough weight to avoid an infeasible solution.

Experiment results

- The Grover-assisted compact genetic algorithm can find the optimal solution.
- TSP is mapped to an Ising model with scales $(n-1)^2$ spins are required, where n is the number of cities, and we designate city 1 to appear first in the route.
- The total cost of quantum state preparation is the sum of the number of qubits and the number of ancilla qubits required, which is $O(n^2-1)$.
- The total quantum complexity of our approach then becomes $O(I\frac{\pi}{4}\sqrt{\frac{n}{t}})$, where I is the circuit depth for Grover iteration at first, n is the number of cities, and t is the number of solutions from oracle.

#Cities	#Qubits	#Ancilla qubits	#CNOT	Circuit depth
3	$(n-1)^2$	$2(n-1)$	$57g$	$2+99g$
4	$(n-1)^2$	$2(n-1)$	$315g$	$1+549g$

n is the number of cities, and g is the number of Grover iterations.

Observation

- To use a quantum algorithm, we have to encode the problem we want to solve into qubits.
- Increasing the number of shots aids in obtaining a probability distribution of results, mitigating stochastic errors, and achieving precise state estimations.
- Quantum computers should be integrated as a subroutine in the broader algorithm to outperform classical computers at certain tasks that are required for optimization algorithms.
- Quantum algorithms we have today only offer modest speed-ups over their classical counterparts.
- Quantum algorithms don’t seem to offer exponential speedups for black box optimization problem. However, it’s possible that there may be some exponential speedup in cases where you know a bit more about the problem.

Q & A