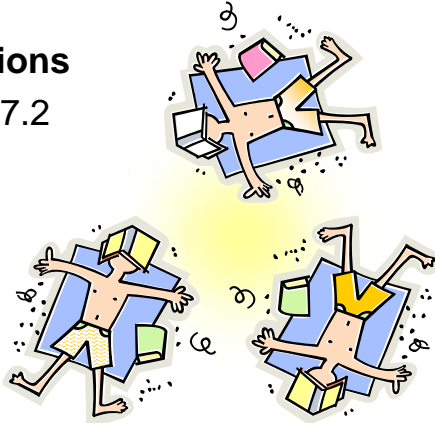




Recurrence Relations

- Readings:
Recurrence Relations
Rosen section 7.1-7.2



Recurrence Relations

- A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms, a_0, a_1, \dots, a_{n-1} .

$$a_n = 5a_{n-1}$$

$$b_n = b_{n-1} - 2b_{n-2} + 100$$

$$c_n = c_{n-3} + c_{n-4} + \log(n) + e^n$$

- A sequence is called a **solution** of a recurrence relation if its terms satisfy the recurrence relation.



Recurrence Relations

- Example (Rosen p.402):
Determine whether $a_n = 3n$, for every nonnegative integer n , is a solution of

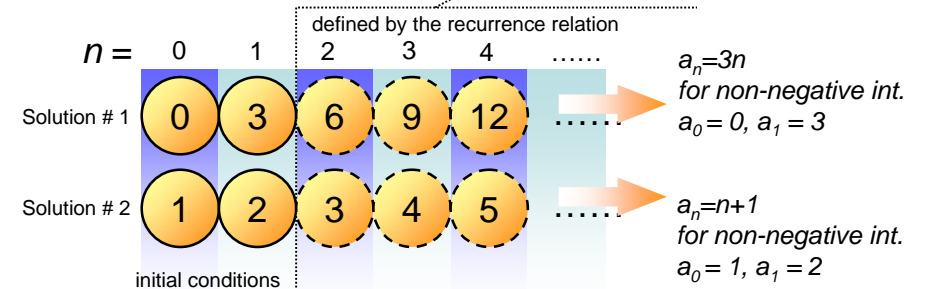
$$a_n = 2a_{n-1} - a_{n-2}; \quad n = 2, 3, \dots$$



Initial Conditions

The **initial conditions** specify the terms that precede the first term where the recurrence relation takes effect.

$$a_n = 2a_{n-1} - a_{n-2}; \quad n = 2, 3, \dots$$



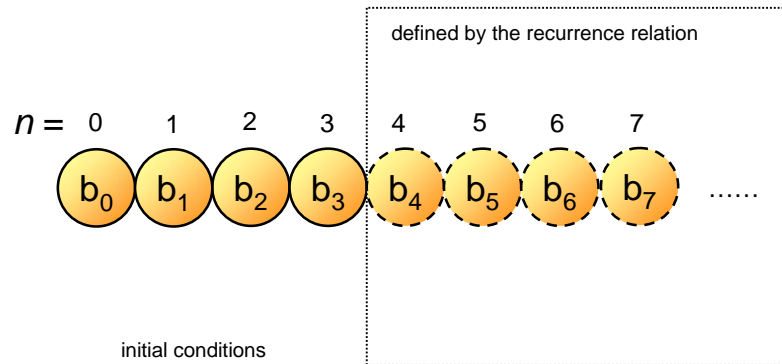


Initial Conditions

In order to find a *unique solution* for every non-negative integers to:

$$b_n = b_{n-2} + b_{n-4}; \quad n = 4, 5, \dots$$

how many terms of b_n needed to be given in the initial conditions?



Modeling with Recurrence Relations

To find solutions for doing a task of a size n

Find a way to:

Construct the solution at the size n from the solution of the same tasks at smaller sizes.



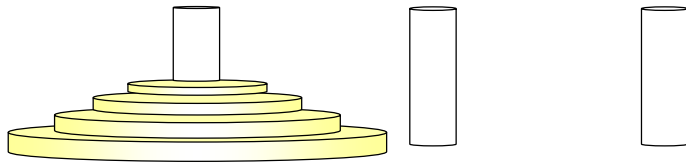
Example: The Tower of Hanoi

Rules:

Move a disk at a time from one peg to another.

Never place a disk on a smaller disk.

The goal is to have all disk on the 2nd peg in order of size.

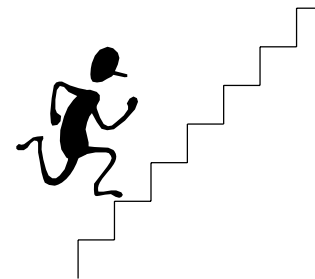


Find H_n , the number of moves needed to solve the problem with n disks.



Example:

A man running up a staircase of n stairs. Each step he takes can cover either 1 or 2 stairs. How many different ways for him to ascend this staircase?





Solving Recurrence Relations

- A **linear homogeneous** recurrence relation can be solved in a systematic way.

Linear homogeneous recurrence relation of degree k with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

where c_1, c_2, \dots, c_k are real numbers and $c_k \neq 0$

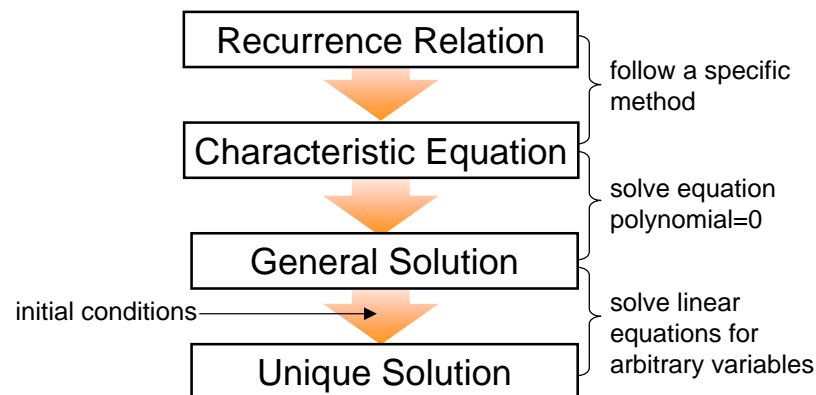


Linear Homogeneous Recurrence Rel.

- $a_n = a_{n-1} + a_{n-2}$
- $H_n = 2H_{n-1} + 1$
- $B_n = nB_{n-1}$
- $f_n = f_{n-1} + f_{n-2}$

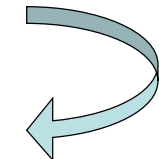


Solving: Linear Homogeneous Recurrence Relations



Solving: Linear Homogeneous Recurrence Relations

- Recurrence Relation:
 $a_n - c_1 a_{n-1} - c_2 a_{n-2} - \dots - c_k a_{n-k} = 0$
- Characteristic equation:
 $r^k - c_1 r^{k-1} - c_2 r^{k-2} \dots - c^k = 0$



Example:

$$a_n = a_{n-1} + 3a_{n-2}$$

$$b_n = 2b_{n-1} - b_{n-2} + 5b_{n-3}$$

$$d_n = d_{n-2} + d_{n-5}$$



Solving: Linear Homogeneous Recurrence Relations

- Let c_1, c_2, \dots, c_k be real numbers. Suppose the **characteristic equation**:

$$r^k - c_1r^{k-1} - c_2r^{k-2} \dots - c_k = 0$$

has k distinct roots r_1, r_2, \dots, r_k . Then a sequence $\{a_n\}$ is a solution of the **recurrence relation**:

$$a_n - c_1a_{n-1} - c_2a_{n-2} - \dots - c_ka_{n-k} = 0$$

if and only if:

$$a_n = \alpha_1r_1^n + \alpha_2r_2^n + \dots + \alpha_kr_k^n$$

for $n = 0, 1, 2, \dots$. Where $\alpha_1, \alpha_2, \dots, \alpha_k$ are constants.



- Example :

What is the solution of the recurrence relation:

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$?



- Example:

What is the solution of the recurrence relation:

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with $a_0 = 2, a_1 = 5$ and $a_2 = 15$?



Repeated Roots

- Suppose the characteristic equation has t distinct roots r_1, r_2, \dots, r_t with multiplicities m_1, m_2, \dots, m_t .
- Solution:

$$a_n = (\alpha_{1,0} + \alpha_{1,1}n + \dots + \alpha_{1,m_1-1}n^{m_1-1})r_1^n + (\alpha_{2,0} + \alpha_{2,1}n + \dots + \alpha_{2,m_2-1}n^{m_2-1})r_2^n + \dots + (\alpha_{t,0} + \alpha_{t,1}n + \dots + \alpha_{t,m_t-1}n^{m_t-1})r_t^n$$



• Example :

What is the solution of the recurrence relation:

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$

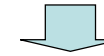
with $a_0 = 1$, $a_1 = -2$ and $a_2 = -1$?



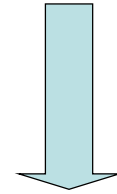
Solving: Linear Nonhomogeneous Recurrence Relations

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n)$$

Associated homogeneous
recurrence relation



$$\{a_n^h\}$$



$$\{a_n^p\}$$

$$\boxed{\{a_n\} = \{a_n^h\} + \{a_n^p\}}$$



Solving: Linear Nonhomogeneous Recurrence Relations

- Key:
 - 1 – Solve for a solution of the associated homogeneous part.
 - 2 – Find a particular solution.
 - 3 – Sum the solutions in 1 and 2
- There is no general method for finding the particular solution for every $F(n)$
- There are general techniques for some $F(n)$ such as *polynomials* and *powers of constants*.



• Example:

Find the solutions of $a_n = 3a_{n-1} + 2n$ with $a_1 = 3$



• Example:

Find the solutions of $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$



Particular Solutions

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \dots + b_1 n + b_0) s^n$$

where b_0, b_1, \dots, b_t and s are real numbers.

When s is **not** a root of the characteristic equation:

The particular solution is of the form:

$$(p_t n^t + p_{t-1} n^{t-1} + \dots + p_1 n + p_0) s^n$$

When s is a root of multiplicity m :

The particular solution is of the form:

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \dots + p_1 n + p_0) s^n$$



• Example:

What form does a particular solution of

$$a_n = 6a_{n-1} - 9a_{n-2} + F(n)$$

have when:

$$F(n) = 3n, F(n) = n3^n, F(n) = n^2 2^n, F(n) = (n^2 + 1)3^n ?$$



• Example :

Find the solution of $a_n = \sum_{k=1}^n k$