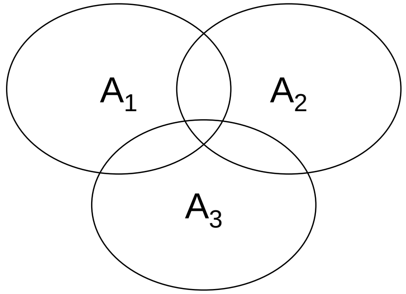
Inclusion-Exclusion Principle

- Readings:

  Rosen section 7.5-7.6

Inclusion-Exclusion

- How many elements are in the union of finite sets?



$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_3 \cap A_1| + |A_1 \cap A_2 \cap A_3|$$

The Principle of Inclusion-Exclusion

- Let $A_1, A_2, \ldots, A_n$ be finite sets. Then

$$\left|A_1 \cup A_2 \cup \cdots \cup A_n\right| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i \le j \le n} |A_i \cap A_j| + \sum_{1 \le i \le j \le k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

- Examples:

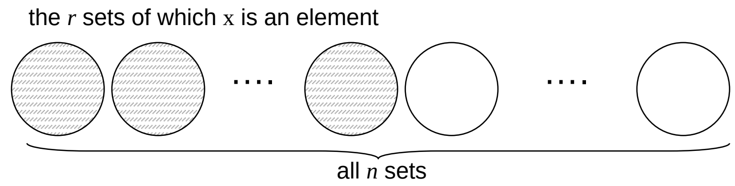
  $\left|A_1 \cup A_2 \cup A_3 \cup A_4\right| =$

Proof: Inclusion–Exclusion Principle

- Showing that an element in the union is counted exactly once.

Let x be an element of exactly r sets.

For example, x is an element of $A_1, A_2, \ldots, A_r$,
But not of $A_{r+1}, A_{r+2}, \ldots, A_n$.



$$\sum_{1\le i\le n}|A_i|$$
$$-\sum_{1\le i1\le i2\le n}|A_{i1}\cap A_{i2}|$$
$$+\sum_{1\le i1\le i2\le i3\le n}|A_{i1}\cap A_{i2}\cap A_{i3}|$$
$$-\cdots+(-1)^{r+1}\sum_{1\le i1\le i2\le\cdots\le ir\le n}|A_{i1}\cap A_{i2}\cap\cdots\cap A_{ir}|$$
$$+\cdots+(-1)^{n+1}|A_1\cap A_2\cap\cdots\cap A_n|$$

An Alternative Form

- Count the number of elements that have none of n properties, $P_1, P_2, \ldots, P_n$
- E.g.:

  P_1: Got an ‘A’ from Physics I

  P_2: Got an ‘A’ from Physics II

  Number of students that never got any ‘A’s from Physics in the first year.

Elements with None of the Properties

- Let A_i be the subset of elements with property P_i.
- Let $N(P_1'P_2'\cdots P_n')$ denote the number of elements with none of the properties $P_1, P_2, .., P_n$

$$N(P_1'P_2'\cdots P_n') = N - |A_1 \cup A_2 \cup \cdots \cup A_n|$$

where N = the total number of elements.

Elements with None of the Properties

$$N(P_1'P_2'\cdots P_n') = N - |A_1 \cup A_2 \cup \cdots \cup A_n|$$

$$N(P_1'P_2'\cdots P_n') = N - \Big(\sum_{1\le i\le n}|A_i| - \sum_{1\le i\le j\le n}|A_i \cap A_j| + \sum_{1\le i\le j\le k\le n}|A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1}|A_1 \cap A_2 \cap \cdots \cap A_n|\Big)$$

$$\boxed{N(P_1'P_2'\cdots P_n') = N - \sum_{1\le i\le n} N(P_i) + \sum_{1\le i\le j\le n} N(P_iP_j) - \sum_{1\le i\le j\le k\le n} N(P_iP_jP_k) + \cdots + (-1)^n N(P_1P_2\cdots P_n)}$$

- Example:

How many solutions does $x_1+x_2+x_3=11$ have,
where x_1 is a non negative integer ≤ 3,
x_2 is a non negative integer ≤ 4,
and x_3 is a non negative integer ≤ 6?

The Number of Onto Functions

- Example:

  How many onto functions are there from a set with 6 elements to a set with 3 elements?

General Result

- Number of onto functions from a set of *m* elements to a set of *n* elements.

- Example:

  How many ways are there to assign five different jobs to four employees if every employee is assigned at least one job?

Derangements

- A ***derangement*** is a permutation of objects that leaves no object in its original position.
- Example:

  Consider a sequence 12345.

  21453

  43512

  42351

Derangements

- The number of derangements of a set with *n* elements, D_n = ?

- <u>Example</u>: "The Hatcheck Problem"

  An employee checks the hats of n people at a restaurant. He forgot to put claim check numbers on the hats. When customers return for their hats, this checker gives hats chosen at random to them.

  What is the probability that no one receives the correct hat?