2143110
DISCRETE MATHEMATICS

Atiwong Suchato, Ph.D.

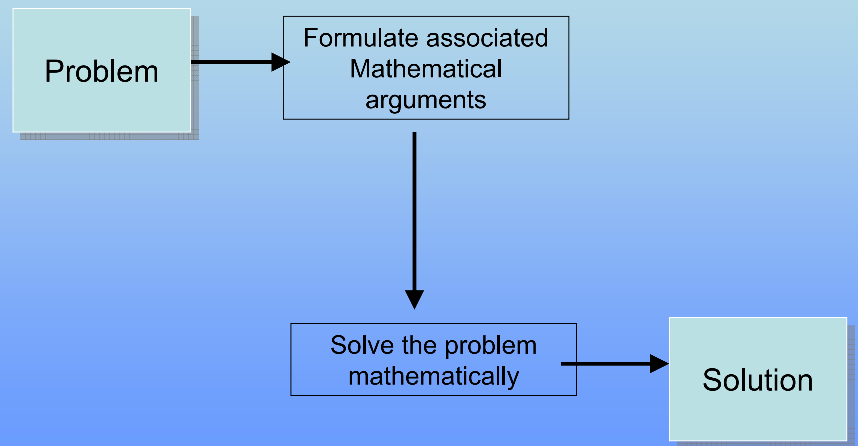
Course Outline

- 4 parts:
- Part1: Logic, Sets, Relations, Functions, and Mathematical Reasoning
- Part2: Graphs and Trees
- Part3: Counting, Recurrence Relations, and Generating Functions
- Part4: Number Theory

Grading

In-class Quiz 1	15%
In-class Quiz 2	15%
In-class Quiz 3	15%
In-class Quiz 4	15%
Final Exam	40%

Why ???



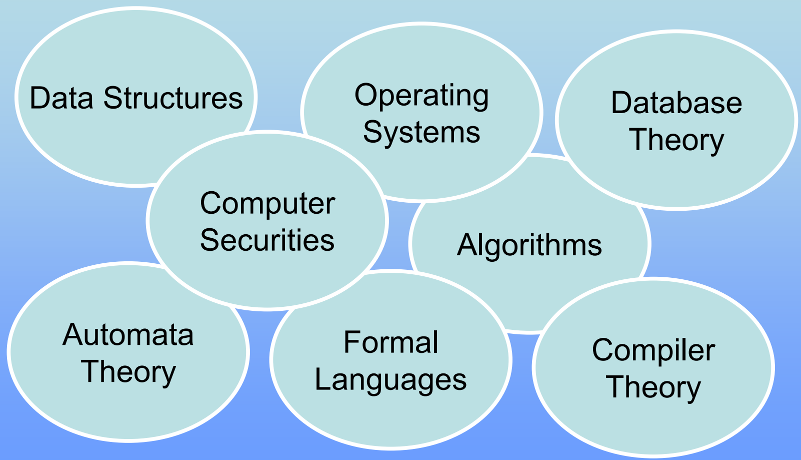
Goals of Discrete Math.

- **Mathematical Reasoning**
  - Read, comprehend, and construct mathematical arguments
- **Combinatorial Analysis**
  - Perform analysis to solve counting problems
- **Discrete Structure**
  - Able to work with discrete structures: sets, graphs, finite-state machines, etc.

Goals of Discrete Math.

- **Algorithmic Thinking**
  - Specify, verify, and analyze an algorithm
- **Applications and Modeling**
  - Apply the obtained problem-solving skills to model and solve problems in computer science and other areas, such as:
    - Business
    - Chemistry
    - Linguistics
    - Geology
    - etc

Gateway to . . .



Foundations of Discrete Math.

- **<u>Logic</u>**
  - Specify the meaning of Mathematical statements
  - Basis of all Mathematical reasoning
- **<u>Sets</u>**
  - Sets are collections of objects, which are used for building many important discrete structures.
- **<u>Functions</u>**
  - Used in the definition of some important structures
  - Represent complexity of an algorithm, and etc.

Readings

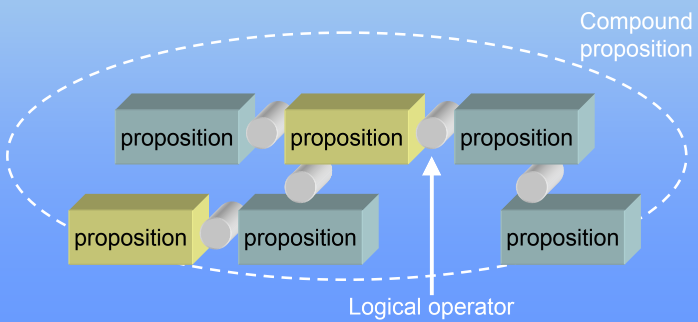
- Rosen: 1.1-1.4, 1.6-1.8, 7.1

Logic

- Rules of logic gives precise meaning to mathematical statements.

Proposition: Building Blocks of Logic

- Proposition =
  - Declarative sentence
  - Either *TRUE* or *FALSE* (not both)



Logical Operators

- Negation (NOT)
- Conjunction (AND)
- Disjunction (OR)
- Exclusive OR (XOR)
- Implication (IF..THEN)
- Biconditional (IF & ONLY IF)

Negation

- The negation of p has opposite truth value to p

p	$\neg p$
T	F
F	T

Atiwong Suchato
Department of Computer Engineering, Chulalongkorn University

Conjunction

- The conjunction of p and q, is true when, and only when, both p and q are true.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Atiwong Suchato
Department of Computer Engineering, Chulalongkorn University
Discrete Mathematics

Disjunction

- The disjunction of p and q, is true when at least one of p or q is true.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Atiwong Suchato
Department of Computer Engineering, Chulalongkorn University
Discrete Mathematics

Exclusive OR

- Exclusive or = OR but NOT both

$$p \oplus q = (p \vee q) \wedge \neg(p \wedge q)$$

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Atiwong Suchato
Department of Computer Engineering, Chulalongkorn University
Discrete Mathematics

Implication

- It is false when p is true and q is false, and true otherwise.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional

- $p \leftrightarrow q$ is true when p and q have the same truth value.
  - Intuitively, $p \leftrightarrow q$ is $(p \rightarrow q) \wedge (q \rightarrow p)$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Contrapositive

- The *contrapositive* of an implication $p \rightarrow q$ is:

$$\neg q \rightarrow \neg p$$

- has the same truth values as $p \rightarrow q$

Converse and Inverse

- The *converse* of an implication $p \rightarrow q$ is:

$$q \rightarrow p$$

- The *inverse* of an implication $p \rightarrow q$ is:

$$\neg p \rightarrow \neg q$$

- DO NOT have the same truth values as $p \rightarrow q$

Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

$p \wedge \neg q \vee r \rightarrow p \leftrightarrow s$

$$\Downarrow$$

$((p \wedge (\neg q) \vee r) \rightarrow p) \leftrightarrow s$

Consistency

- Translating natural language to logical expressions is essential to specifying system spec.
- Specifications are ***"consistent"*** when they do not conflict with one another. i.e.:

> There must be an assignment of truth values to every expression that make all the expression true.

Consistency

- Whenever the system is being upgraded, users cannot access the file system.
- If users can access the file system, they can save new files.
- If users cannot save new files, the system is not being upgraded.

Consistency

- Whenever the system is being upgraded, users cannot access the file system. $p \rightarrow \neg q$
- If users can access the file system, they can save new files. $q \rightarrow r$
- If users cannot save new files, the system is not being upgraded. $\neg r \rightarrow \neg p$

p	q	r	$p \rightarrow \neg q$	$q \rightarrow r$	$\neg r \rightarrow \neg p$
T	F	T	T	T	T

These spec. are consistent.

Tautology, Contradiction, & Contingency

- A compound proposition that is always *true* is called a ***"tautology"***.
- A compound proposition that is always *false* is called a ***"contradiction"***.
- If neither a tautology nor a contradiction, it is called a ***"contingency"***.

Logical Equivalences

The propositions p and q are called "**logical equivalent**" ($p \equiv q$) if $p \leftrightarrow q$ is a tautology

Showing Logically Equivalent propositions

1 Show that the truth values of these propositions are always the same.

$\rightarrow$ Construct truth tables.

Showing Logically Equivalent propositions

- Example (Rosen p22):
  Show that $p \rightarrow q \equiv \neg p \vee q$

p	q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Logically Equivalent

Showing Logically Equivalent propositions

1. Show that the truth values of these propositions are always the same.
2. Use series of established equivalences.

Logical Equivalences

- Distributive Laws

  $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

  $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- De Morgan's Laws

  $\neg (p \wedge q) \equiv \neg p \vee \neg q$

  $\neg (p \vee q) \equiv \neg p \wedge \neg q$
- More can be found in Rosen p.24

Showing Logically Equivalent propositions

- Example (Rosen p25):

  Show that $\neg (p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q)$ *De Morgan's*

$\equiv \neg p \wedge (\neg(\neg p) \vee \neg q)$ *De Morgan's*

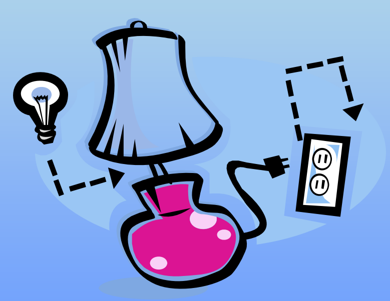
$\equiv \neg p \wedge (p \vee \neg q)$ *Double negative*

$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$ *Distributive*

$\equiv F \vee (\neg p \wedge \neg q)$

$\equiv \neg p \wedge \neg q$

Predicate Logic



Predicate Logic

- In Propositional Logic, 'the atomic units' are propositions.
- E.g.:
  – *p*: John goes to school., *q*: Mary goes to school.
- In Predicate Logic, we look at each proposition as the combination of ***variables*** and ***predicates*** .
- E.g.:
  – X goes to school, where X can be John or Mary.

Predicate Logic

- The statement "*x* go to school" has two parts:
  Variable "*x*"
  The predicate "go to school"
- This statement can be denoted by *P(x)*, where *P* denotes the predicate "go to school".
- *P(x)* is said to be the value of the propositional function *P* at *x*.
- Once a value has been assigned to the variable *x*, the statement *P(x)* becomes a proposition and has a truth value.
- E.g: P(John) and P(Mary) have truth values.

Creating propositions from a propositional function

1. Assign values to all variables in a propositional function.
2. Use "Quantification"

Universal Quantifier

- $\forall x P(x)$ (read "for all x P(x)") denotes:

  P(x) is true for all values *x* in the universal of discourse.

- $\forall x P(x)$ is the same as:

  $$P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$$

  When all elements in the universe of discourse can be listed as $(x_1, x_2, \ldots, x_n)$

Universal Quantifier

- <u>Example</u> (Rosen p.31):
- What is the truth value of $\forall xP(x)$, when $P(x)$ is $x^2 \geq x$ and the universe of discourse consists of:
  - all real numbers?
  - all integers?

  Since $x^2 \geq x$ only when $x \leq 0$ or $x \geq 1$, $\forall xP(x^2 \geq x)$ is false if the universe consists of all real numbers. However, it is true when the universe consists of only the integers.

Existential Quantifier

- $\exists xP(x)$ (read "for some x P(x)") denotes:

  There exists an element *x* in the universe of discourse that *P(x)* is true.

- $\exists xP(x)$ is the same as:

  $$P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$$

  When all elements in the universe of discourse can be listed as $(x_1, x_2, \ldots, x_n)$

Existential Quantifier

- <u>Example</u> (Rosen p.32):
- What is the truth value of $\exists xP(x)$ where $P(x)$ is the statement $x^2 > 10$, and the universe of discourse consists of the positive integers not exceeding 4?

  Since the elements in the universe can be listed as {1,2,3,4}, $\exists xP(x)$ is the same as $P(1) \vee P(2) \vee P(3) \vee P(4)$. There for $\exists xP(x)$ is true since $P(4)$ is true.

Negations

$$\neg\forall xP(x) \equiv \exists x\, \neg P(x)$$

$$\neg\exists xP(x) \equiv \forall x\, \neg P(x)$$

Negation of
"Every ICE students loves Discrete math." is
"There is an ICE student who does not love Discrete math."
Negation of
"Some student in this class get 'A'." is
"None of the students in this class get 'A'."

Nested Quantifiers

- Quantifiers that occur within the scope of other quantifiers.
- E.g.:
  $\forall x \forall y((x > 0) \wedge (y < 0) \rightarrow (xy < 0))$

Atiwong Suchato
Department of Computer Engineering, Chulalongkorn University

Nested Quantifiers

Statement	… is TRUE when
$\forall x \forall y$ P(x,y) $\forall y \forall x$ P(x,y)	P(x,y) is true for every pair of x,y
$\forall x \exists y$ P(x,y)	For every x, there is a y for which P(x,y) is true.
$\exists x \forall y$ P(x,y)	There is an x for which P(x,y) is true for every y.
$\exists x \exists y$ P(x,y) $\exists y \exists x$ P(x,y)	There is a pair x,y for which P(x,y) is true.

Atiwong Suchato
Department of Computer Engineering, Chulalongkorn University
Discrete Mathematics

Logic: Key Terms

- Proposition
- Truth value
- Negation
- Logical Operator
- Compound proposition
- Truth table
- Disjunction
- Conjunction
- Exclusive or
- Implication
- Inverse
- Converse
- Contrapositive
- Biconditional
- Bit operations
- Tautology
- Contradiction
- Contingency
- Consistency
- Logical equivalence
- Predicate
- Propositional function
- Universe of discourse
- Existential quantifier
- Universal quantifier

Atiwong Suchato
Department of Computer Engineering, Chulalongkorn University
Discrete Mathematics