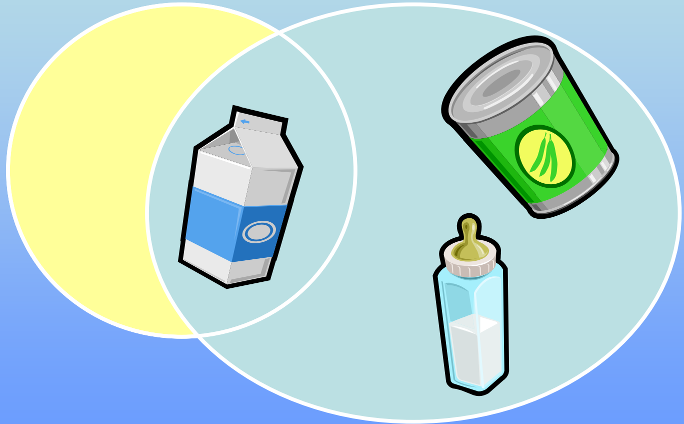
Sets



Sets

- A set is an unordered collection of objects.
- Objects in a set are called “members” or “elements” of that set.
- Two sets are equal $\leftrightarrow$ they have the same elements

- Are $\{1,2,3\}$ and $\{3,2,1\}$ equal?
- Are $\{0,1,2\}$ and $\{0,0,0,1,1,2\}$ equal?

Set Builder Notation

- Stating the properties that all elements must have to be members.

O = {x | x is a prime number less than 100}
R = {x | x is a real number}
U = {x | x is any of the objects under consideration}

Subset

$$A \subseteq B \leftrightarrow \forall x \, (x \in A \rightarrow x \in B)$$

Proper Subset

$$A \subset B \leftrightarrow (A \subseteq B) \wedge (A \neq B)$$

For any set S, “$\varnothing \subseteq S$” and “$S \subseteq S$”

Proof: $\varnothing \subseteq S$ and $S \subseteq S$

- Show that $\forall x(x\in\varnothing \rightarrow x\in S)$
  - Since $x\in\varnothing$ is always false, then $x\in\varnothing \rightarrow x\in S$ is always true no matter what x is.
- Show that $\forall x(x\in S \rightarrow x\in S)$
  - Since $p \rightarrow p$ is a tautology the $x\in S \rightarrow x\in S$ is true no matter what.

Atiwong Suchato
Department of Computer Engineering, Chulalongkorn University

Cardinality

- For a set *S*, if there are exactly *n* distinct elements in *S*, where *n* is a nonnegative interger, we say that *S* is a *finite set* and that *n* is the *cardinality* of *S* ($|S|=n$)
- A set is "infinite" if it is not finite.

Atiwong Suchato
Department of Computer Engineering, Chulalongkorn University

Power Set

- Given a set *S*, the power set of *S*, ***P(S)***, is the set of all subsets of *S*
- If *S* has *n* elements, then *P(S)* has 2^n elements.

- Examples (Rosen p.82):

S	P(S)
{0,1,2}	{∅,{0},{1},{2},{0,1},{0,2},{1,2},{0,1,2}}
∅	{∅}
{∅}	{∅,{∅}}

Atiwong Suchato
Department of Computer Engineering, Chulalongkorn University

Ordered n–tuple

- The *ordered n-tuple* $(a_1,a_2,..,a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element,…, and an as its n^{th} element.

Two ordered n-tuples are equal ↔ each corresponding pair of their elements is equal

Atiwong Suchato
Department of Computer Engineering, Chulalongkorn University

Cartesian Products

$A \times B = \{ (a,b) \mid a \in A \wedge b \in B\}$

$A_1 \times A_2 \times \dots \times A_n = \{ (a_1,a_2,\dots,a_n) \mid a_i \in A_i \text{ for } i=1,2,..,n\}$

- Examples:
- What is the Cartesian product AxBxC, where A={0,1}, B={j,k}, C={x,y,z}?

AxBxC={(0,j,x),(0,j,y),(0,j,z),(0,k,x),(0,k,y),(0,k,z),
(1,j,x),(1,j,y),(1,j,z),(1,k,x),(1,k,y),(1,k,z)}

Using Set Notation with Quantifiers

- Specify the universe of discourse .
- E.g.:

$\forall x \in \mathbf{R}(x^2 \geq 0)$

means "for every real number $x^2 \geq 0$"

which is true.

Set Operations

- Union ($\cup$)
- Intersection ($\cap$)
- Difference ($-$)
- Complement ($'$)
- Symmetric difference ($\oplus$)

Symmetric Difference

- $A \oplus B$ is the set containing those elements in *either A or B* but *NOT in both A and B*.

Example:
A = {1,3,5}, B = {1,2,3}, $A \oplus B$ = {2,5}

Principle of Inclusion–Exclusion

$$|A \cup B| = |A|+|B|-|A \cap B|$$

More general (Chapter 6):

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \Sigma|A_i| - \Sigma|A_i \cap A_j| + \Sigma|A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1}|A_1 \cap A_2 \cap \ldots \cap A_n|$$

Set Identities

- Distributive Laws

  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

- De Morgan's Laws

  $(A \cup B)' = A' \cap B'$

  $(A \cap B)' = A' \cup B'$

- More can be found in Rosen p.89

Showing that two sets have the same elements

1 Show that each set is a subset of the other.

2 Use set builder notation and logical equivalences.

3 Build membership tables.

4 Use set identities.

Proving Set Equality

Showing that each is a subset of the other

- Example (Rosen p.89): Prove that $(A \cap B)' = A' \cup B'$

1) Suppose $x \in (A \cap B)'$. So, $x \notin A \cap B$
   Then, $\neg((x \in A) \wedge (x \in B))$ is true.
2) De Morgan's $\Rightarrow \neg(x \in A) \vee \neg(x \in B)$ is true.
   Then, $x \in A' \vee x \in B'$
3) Definition of Union $\Rightarrow x \in A' \cup B'$
   $x \in (A \cap B)' \rightarrow x \in A' \cup B'$
   This shows $(A \cap B)' \subseteq A' \cup B'$

Proving Set Equality

Showing that each is a subset of the other

- Example (Rosen p.89): Continued

4) Suppose $x \in A' \cup B'$.
Definition of Union $\Rightarrow x \in A' \vee x \in B'$
$\neg(x \in A) \vee \neg(x \in B)$ is true.

5) Then, $\neg(x \in A \cap B)$ is true.
$\therefore x \notin A \cap B$. So, $x \in (A \cap B)'$.

6) $x \in A' \cup B' \rightarrow x \in (A \cap B)'$
This shows $A' \cup B' \subseteq (A \cap B)'$

Atiwong Suchato
Department of Computer Engineering, Chulalongkorn University

Proving Set Equality

Showing that each is a subset of the other

- Example (Rosen p.89): Continued

3) $(A \cap B)' \subseteq A' \cup B'$
and $\rightarrow (A \cap B)' = A' \cup B'$

6) $A' \cup B' \subseteq (A \cap B)'$

Atiwong Suchato
Department of Computer Engineering, Chulalongkorn University

Proving Set Equality

Using set builder notation and logic equivalances

- Example (Rosen p.89): Prove that $(A \cap B)' = A' \cup B'$

$$
\begin{aligned}
(A \cap B)' &= \{ x \mid x \notin A \cap B \} \\
&= \{ x \mid \neg(x \in A \cap B) \} \\
&= \{ x \mid \neg((x \in A) \wedge (x \in B)) \} \\
&= \{ x \mid (x \notin A) \vee (x \notin B) \} \\
&= \{ x \mid (x \in A') \cup (x \in B') \} \\
&= \{ x \mid x \in A' \cup B' \} \\
&= A' \cup B'
\end{aligned}
$$

Atiwong Suchato
Department of Computer Engineering, Chulalongkorn University

Proving Set Equality

Using membership table

- Example Prove that $(A \cap B)' = A' \cup B'$

A	B	A′	B′	$A' \cup B'$	$(A \cap B)$	$(A \cap B)'$
0	0	1	1	1	0	1
0	1	1	0	1	0	1
1	0	0	1	1	0	1
1	1	0	0	0	1	0

Atiwong Suchato
Department of Computer Engineering, Chulalongkorn University

Proving Set Equality

Use set identities

- Example (Rosen p.91):
  Show that $(A \cup (B \cap C))' = (C' \cup B') \cap A'$

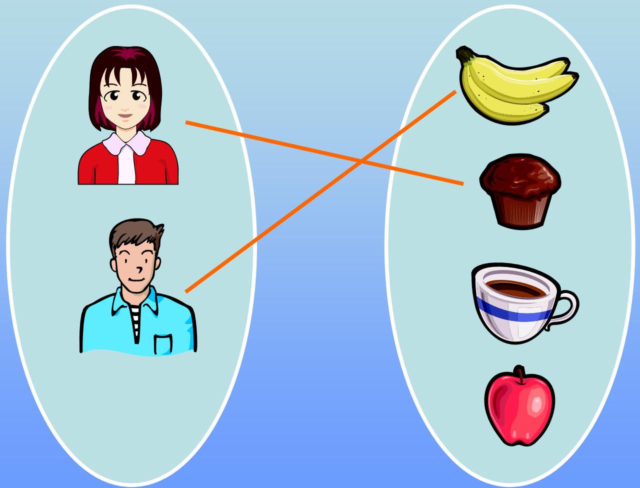
$$
\begin{aligned}
(A \cup (B \cap C))' &= A' \cap (B \cap C)' \\
&= A' \cap (B' \cup C') \\
&= (B' \cup C') \cap A' \\
&= (C' \cup B') \cap A'
\end{aligned}
$$

Generalized Union and Intersection

$$A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$$

$$A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i$$

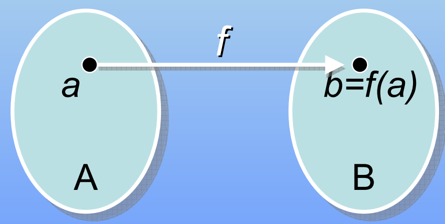
Functions



Functions

Definition:

- A function f from A to B is an assignment.
- assigns exactly one element of B to each of A



A: Domain
B: Codomain
b is *the image* of a.
a is *a pre-image* of b.
Range of f is the set of all images.

- Function cannot be “one-to-many”.
- $\forall a \in A$, $f(a)$ must be assigned to some b.

Adding and Multiplying Functions

- Two real-valued functions *with the same domain* can be added and multiplied.

f_1, f_2 are functions from A to $\boldsymbol{R}$
$\rightarrow f_1+f_2$ and f_1f_2 are also functions from A to $\boldsymbol{R}$.

$$(f_1+f_2)(x) = f_1(x)+f_2(x)$$
$$(f_1f_2)(x) = f_1(x)f_2(x)$$

Adding and Multiplying Functions

- Example (Rosen p.99):
- f_1, f_2 are functions from $\boldsymbol{R}$ to R. $f_1(x)=x^2$, $f_2(x)=x-x^2$. What are the functions f_1+f_2 and f_1f_2?

(f1+f2)(x) = f1(x)+f2(x) = x2 + x - x2 = x

(f1f2)(x) = f1(x)f2(x) = x2 (x - x2) = x3 – x4

One-to-one Functions

A function f is *one-to-one* or *injective*
$\leftrightarrow \forall x \forall y\ (\ f(x)=f(y) \rightarrow x=y\ )$

Examples (Rosen p.100)
Determine whether these functions are one-to-one.

$f_1(x) = x^2$ from the set of integers to the set of integers
Since $f(1) = f(-1) = 1$, $f_1(x)$ is not one-to-one.

$f_2(x) = x+1$
$x+1 \neq y+1$ when $x \neq y$, then $f_2(x)$ is one-to-one.

Conditions Guaranteeing One-to-one

- Strictly increasing function:

$\forall x\ \forall y\ (\ (x<y) \rightarrow (f(x)<f(y))\ )$

- Strictly decreasing function:

$\forall x\ \forall y\ (\ (x<y) \rightarrow (f(x)>f(y))\ )$

where the universe of discourse = domain of f

Strictly increasing function
or
Strictly decreasing function
$\rightarrow$ one-to-one

Onto Functions

A function f is *onto* or *surjective*
$\leftrightarrow \forall y \exists x\ (f(x) = y)$

Examples (Rosen p.101)
Determine whether these functions are onto.

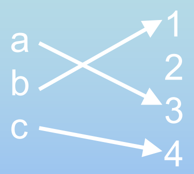
$f_1(x) = x^2$ from the set of integers to the set of integers
No, since there is no integer x that $f_1(x)=-1$

$f_2(x) = x+1$
Yes, for every $f_2(x)=y$, there is an integer $x=y-1$

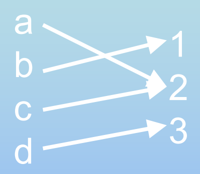
One-to-one Correspondence

- *One-to-one* AND *Onto*
- Also called *"bijection"*

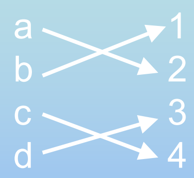
Examples



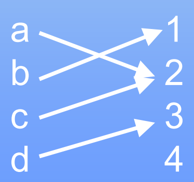
1-to-1, not onto



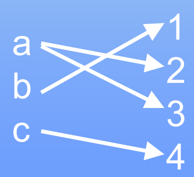
not 1-to-1,onto



1-to-1,onto



neither 1-to-1,nor onto



not a function

Inverse Functions

- Let *f* be a *one-to-one correspondent* function from *A* to *B*.
- $f^{-1}(b)$ assigns to *b*, belonging to *B*, the unique element *a*, belonging to *A*, such that $f(a)=b$.

$$f^{-1}(b)=a \leftrightarrow f(a)=b$$

A function that is *NOT one-to-one correspondent* is *NOT invertible*.

Composite Functions

- $(f \bullet g)(a) = f(\ g(a)\ )$
- $f \bullet g$ cannot be defined unless the range of g is a subset of the domain of f.
- If f is a one-to-one correspondent function from A to B

$$(f^{-1} \bullet f)(a) = a, \quad a \in A$$
$$(f \bullet f^{-1})(b) = b, \quad b \in B$$

Atiwong Suchato
Department of Computer Engineering, Chulalongkorn University

Some Important Functions

- Floor function $\lfloor\ \rfloor$

$\lfloor x \rfloor$ = the largest integer $\leq x$

- Ceiling function $\lceil\ \rceil$

$\lceil x \rceil$ = the smallest integer $\geq x$

$\lfloor 1/2 \rfloor =$ $\quad$ $\lfloor -1/2 \rfloor =$ $\quad$ $\lfloor 1 \rfloor =$

$\lceil 1/2 \rceil =$ $\quad$ $\lceil -1/2 \rceil =$ $\quad$ $\lceil 1 \rceil =$

Atiwong Suchato
Department of Computer Engineering, Chulalongkorn University

Examples

- <u>Example </u>(Rosen p.106):
- Each byte is made up of 8 bits. How many bytes are required to encoded 100 bits of data?

$\lceil 100/8 \rceil = \lceil 12.5 \rceil = 13$ bytes

Atiwong Suchato
Department of Computer Engineering, Chulalongkorn University

Factorial Function

- $f(n) = n!$ is the product of the first n positive integers, so that

$$f(n) = 1 \cdot 2 \cdot \ldots \cdot (n-1) \cdot n$$
$$\text{and } f(0) = 0! = 1$$

Atiwong Suchato
Department of Computer Engineering, Chulalongkorn University

Sets: Key Terms

- Set
- Element
- Member
- Empty/Null set
- Universal set
- Venn diagram
- Set equality
- Subset
- Proper subset
- Finite set
- Infinite set
- Cardinality
- Power set
- Union
- Intersection
- Difference
- Complement
- Symmetric difference
- Membership table

Functions: Key Terms

- Function
- Domain
- Codomain
- Image
- Pre-image
- Range
- Onto / Surjection
- One-to-one / Injection
- One-to-one correspondence / bijection
- Inverse
- Composition
- Floor function
- Ceiling function
- Factorial