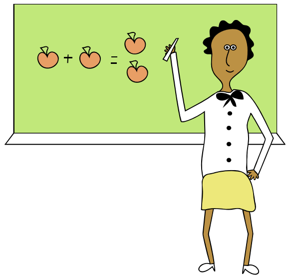
Relations

- Rosen: Section ___



Relations

- A (binary) relation form A to B is a subset of AxB
- A *relation on the set A* is a *relation from A to A*
- A function from A to B is a relation from A to B
- Examples:

  $R_1 = \{(1,1),(1,2),(2,1),(2,3)\}$

  $R_2 = \{(a,b) \mid a = b \text{ or } a = -b\}$

  a and b are integers

Properties of Relations

- R on the set A is *reflexive* $\leftrightarrow \forall a\ ((a,a) \in R)$

Example: Consider relations on {1,2,3,4}

R must contain (1,1),(2,2),(3,3),(4,4)

R1 = {(1,1),(1,2),(1,3),(2,2),(3,3),(4,1),(4,4)}

R2 = {(1,1),(2,1),(2,3),(3,1),(3,2),(3,3),(3,4),(4,4)}

Symmetric and Antisymmetric

- R on a set A is *symmetric*

  $\leftrightarrow \forall a \forall b((a,b) \in R \rightarrow (b,a) \in R)$
- R on a set A is *antisymmetric*

  $\leftrightarrow \forall a \forall b(((a,b)\in R \wedge (b,a)\in R) \rightarrow (a=b))$
- These two are NOT opposite.

Symmetric and Antisymmetric

- *Symmetric* $\leftrightarrow \forall a \forall b((a,b) \in R \rightarrow (b,a) \in R)$
- *Antisym.* $\leftrightarrow \forall a \forall b(((a,b) \in R \wedge (b,a) \in R) \rightarrow (a=b))$

	Sym	Antisym
$R_1 = \{(1,1),(1,2),(2,1)\}$		
$R_2 = \{(1,1),(1,2)\}$		
$R_3 = \{(a,b) \mid a = b \}$ (on Int.)		
$R_4 = \{(2,1)\}$		
$R_5 = \{(a,b) \mid a + b \leq 3\}$ (on Int.)		

Transitive Relations

- R on a set A is *transitive*

  $\leftrightarrow \forall a \forall b \forall c(((a,b) \in R \wedge (b,c) \in R) \rightarrow (a,c) \in R)$

Example:

$R_1 = \{(1,2),(2,3),(1,3),(1,4)\}$
$R_2 = \{(1,1),(1,2),(1,3),(2,4)\}$
$R_3 = \{(a,b) \mid a < b\}$

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Combining Relations

- Since a relation is a set, we can apply all set operators to relations.
- Example

  $R_1 = \{(1,1),(2,2),(3,3)\}$,
  $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$

  $R_1 \cap R_2 = \{(1,1)\}$
  $R_1 - R_2 = \{(2,2),(3,3)\}$

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Composite Relations

- R is a relation from A to B
- S is a relation from B to C
- $S \circ R = \{(a,c) \mid a \in A, c \in C$, and there exists $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S\}$

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Composite Relations

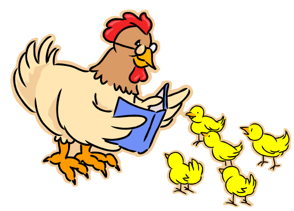
- <u>Example</u>

  R is a relation from {1,2,3} to {1,2,3,4} with R={(1,1),(1,4),(2,3),(3,1),(3,4)} and S is a relation from {1,2,3,4} to {0,1,2} with S={(1,0),(2,0),(3,1),(3,2),(4,1)}.

  What is the composite of R and S?

Methods of Proof

- <u>Readings:</u>



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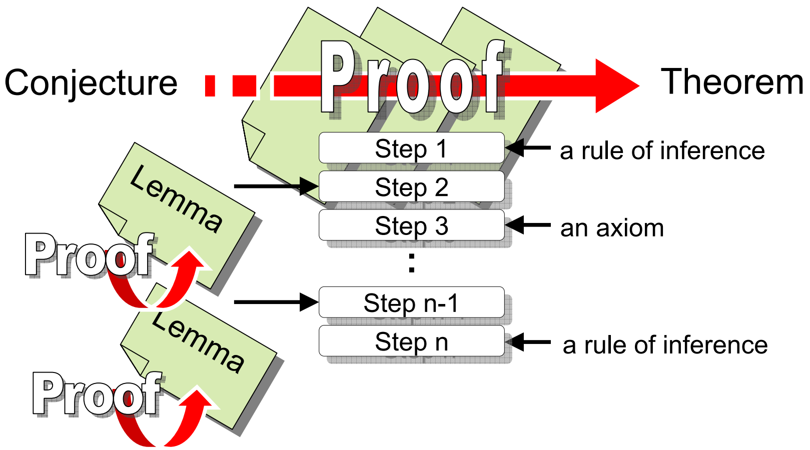
Proof Mechanisms



Rules of Inference

- Provide justification of the steps used to show that *a conclusion follows a set of hypotheses*.
- Each uses *a tautology* as its basis.
- E.g.:

  <u>The *law of detachment* or *Modus ponens*</u>

  $$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}$$

  (Based on $(p \wedge (p \rightarrow q)) \rightarrow q$)

Rules of Inference

Addition	$\underline{p\quad\quad.}$ $\therefore p \vee q$	Modus tollens	$\neg q$ $\underline{p \to q}$ $\therefore \neg p$
Simplification	$\underline{p \wedge q}$ $\therefore p$	Hypothetical syllogism	$p \to q$ $\underline{q \to r}$ $\therefore p \to r$
Conjunction	p $\underline{q\quad\quad.}$ $\therefore p \vee q$	Disjunction syllogism	$p \vee q$ $\underline{\neg p\quad\quad.}$ $\therefore q$
Modus ponen	p $\underline{p \to q}$ $\therefore q$	Resolution	$p \vee q$ $\underline{\neg p \vee r}$ $\therefore q \vee r$

Rules of Inference

- Example:

  If it rains today, we will not have a barbecue today.

  If we do not have a barbecue today, we will have it tomorrow

  Therefore, if it rains today, then we will have a barbecue tomorrow.

  Which rule of inference is used?

Rules of Inference

- Example:

  If it floods today, Chula will close.

  Chula is not closed today.

  Therefore, it did not flood today.

  Which rule of inference is used?

Valid Arguments

- An argument is called ***valid*** if whenever all the hypotheses are true, the conclusion is also true.

  Showing that $(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \to q$ is true.

Valid Arguments

- Example:

h_1: If you send me an email, I will finish writing this program.

h_2: If you do not send me an email, I will go to bed early.

h_3: If I go to bed early, I will wake up feeling refreshed.

Lead to?: *If I do not finish writing program, then I will wake up feeling refreshed.*

Valid Arguments

- Example

Show that $(p \wedge q) \vee r$ and $r \rightarrow s$ imply $p \vee s$

Rules of Inference: Quantified Statements

Universal Instantiation	$\underline{\forall x P(x)}$ $\therefore P(c)$
Universal Generalization	$\underline{P(c) \text{ for an arbitrary } c}$ $\therefore \forall x P(x)$
Existential Instantiation	$\underline{\exists x P(x)}$ $\therefore P(c)$ for some element c
Existential Generalization	$\underline{P(c) \text{ for some element } c}$ $\therefore \exists x P(x)$

Rules of Inference: Quantified Statements

- Example

Show that:

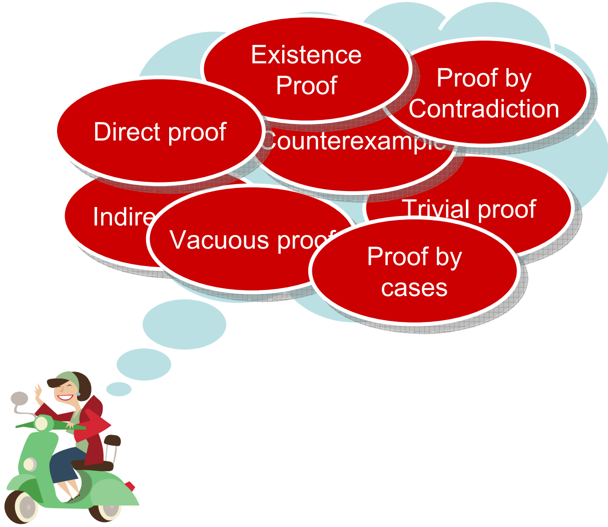
A student in this class has not read the book.

Everyone in this class passed the first exam.

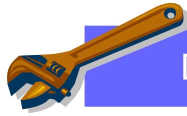
imply:

Someone who passed the first exam has not read the book.

Methods of Proving Theorems



Proving p→q



- Direct Proof
- Indirect Proof
- Vacuous Proof
- Trivial Proof
- Proof by Contradiction

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Proving p→q

- Direct Proof
- Indirect Proof
- Vacuous Proof
- Trivial Proof
- Proof by Contradiction

Direct Proof: Show that if **p** is true, **q** must be true.

Indirect Proof: $p \rightarrow q \equiv \neg q \rightarrow \neg p$

Show that if **¬q** is true, **¬p** must be true.

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Proving p→q

- <u>Example</u> :

  Show that "If *n* is an odd integer, n^2 is an odd integer"

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Proving p→q

- Example :
  Show that "If n is an integer and n^2 is odd, then n is odd."

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Proving p→q

Direct Proof	
Indirect Proof	
Vacuous Proof	Show that p is false. So, p→q is always true.
Trivial Proof	Show that q is true. So, p→q is always true.
Proof by Contradiction	

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Proving p→q

- Example
  P(n) = "If $n > 1$,then $n^2 > n$"
  Show that P(0) is true.
- Example
  P(n) = "If a and b are positive integers with $a \geq b$, then $a^n \geq b^n$"
  Show that P(0) is true.

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Proof by Contradiction

- Proof by Contradiction
  - Suppose we want to prove a statement ***s***
  - Start by assuming $\neg$ ***s*** is true.
  - Show that $\neg$ ***s*** implies a contradiction. ($\neg s \rightarrow F$)
  - Then, $\neg$ ***s*** must be false (or ***s*** must be true).

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Proof by Contradiction

- Example:
  Show that at least 10 of any 64 days chosen must fall on the same day of the week.

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Proof $p \rightarrow q$ by Contradiction

- Proof by Contradiction
  - Start by assuming $\neg(p \rightarrow q)$ is true.
  - That means $p \wedge \neg q$ is true.
    (since $\neg(p \rightarrow q) \equiv \neg(\neg p \vee q) \equiv p \wedge \neg q$)
  - Show that $p \wedge \neg q$ is a contradiction
  - Then, $\neg(p \rightarrow q)$ must be false
    (or $(p \rightarrow q)$ must be true).

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Proving $p \rightarrow q$

- Example:
  Prove that "If n is an integer and n^3+5 is odd, then n is even". Using:
  (a) an indirect proof.
  (b) a proof by contradiction.

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Proof by Cases

- Prove an implication of the form:

$$(p_1 \vee p_2 \vee \ldots \vee p_n) \rightarrow q$$

by proving that:

$$p_i \rightarrow q, \ i = 1,2, \ldots, n$$

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Proof by Cases

- Example:
  Show that $|xy| = |x||y|$, where x and y are real numbers.

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Proof of $p \leftrightarrow q$

- Since $(p \leftrightarrow q) \leftrightarrow (p \rightarrow q) \land (q \rightarrow p)$, then *prove both $p \rightarrow q$ and $q \rightarrow p$*
- Equivalent propositions ($p_1 \leftrightarrow p_2 \leftrightarrow \ldots \leftrightarrow p_n$) are proven by *proving $p_1 \rightarrow p_2$, $p_2 \rightarrow p_3$, …, $p_n \rightarrow p_1$*

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Equivalent Propositions

- Example
  Show that these statements are equivalent:
  p_1: n is an even integer.
  p_2: $n - 1$ is an odd integer.
  p_3: n^2 is an even integer.

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Proof of Proposition Involving Quantifiers

- Existence proofs: A proof of $\exists x P(x)$
- *Constructive existence proof:*
  – Find an element c such that $P(c)$ is true.
- *Non-constructive existence proof:*
  – Do not find an element c such that $P(c)$ is true, but use some other ways.

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Existence Proofs

- <u>Example</u> :

  Show that $\exists x \exists y$ (x^y is rational.) where x and y are irrational.

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Proof of Proposition Involving Quantifiers

- <u>Uniqueness proofs</u>: showing that there is a unique element x such that P(x).

  1) *Existence*:

     Show that $\exists x P(x)$

  2) *Uniqueness*:

     Show that if $y \neq x$, P(y) is false.

- is the same as proving:

$$\exists x(P(x) \wedge \forall y(y \neq x \rightarrow \neg P(y)))$$

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Uniqueness Proofs

- <u>Example:</u>

  Show every integer has a unique additive inverse. (If p is an integer, there exists a unique integer q such that p+q = 0.)

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Counterexamples

- Show that $\forall x P(x)$ is false.

- <u>Example</u>:

  "Every positive integer is the sum of the squares of three integers" ?

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