Chapter 4
Manipulator Kinematics

4.1 Introduction

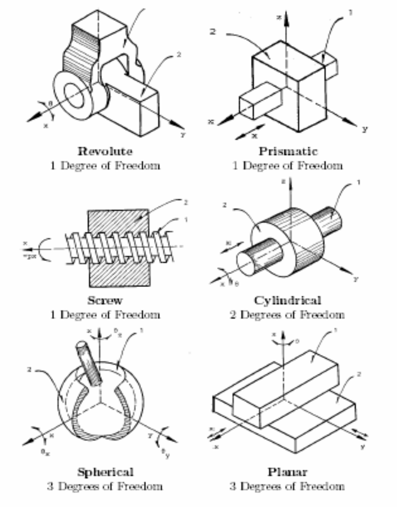
4.2 Coordinate Transformation

4.3 Forward Kinematics

4.4 Inverse Kinematics

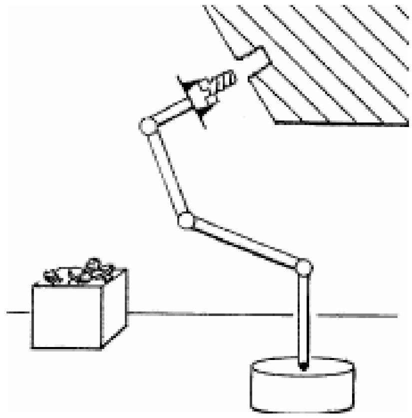
Manipulator

Joint



Gears: Motion from Rotation

Achieving Tasks



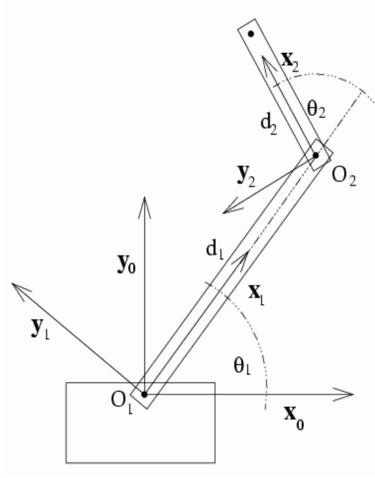
แขนกลประกอบด้วย link และ joint เราเรียกส่วนปลายของแขนกลที่มักติดกับอุปกรณ์ในการทำงานว่า end effector

เมื่อต้องการให้ end effector ไปอยู่ใน position และ orientation ที่ต้องการ เราต้องสามารถกำหนดพารามิเตอร์ของข้อต่อที่เหมาะสมได้

Coordinate Transformation

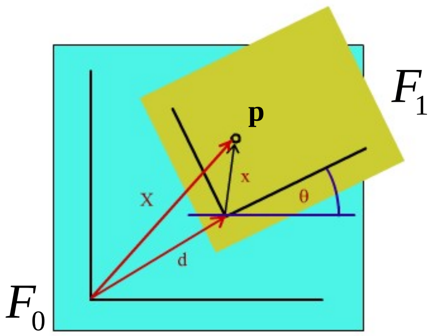
- Frame
- Rotation in 2D
- Rotation in 3D
- Motion in 2D and 3D
- Transformation Matrix

Frame



ท่อนแขนแต่ละท่อนมี frame ของตัวเอง เมื่อต้องการหาพิกัดของจุดที่กำหนดบนท่อนแขนหนึ่งต้องรู้ความสัมพันธ์ของการเคลื่อนที่ระหว่างท่อนแขน

Coordinate Transformation



จุด p มี coordinates ใน F_1 คือ x

สมมุติว่า coordinates ของมันใน F_0 เขียนได้เป็น

$$\mathbf{X} = [A]\mathbf{x} + \mathbf{d}$$

พิจารณาสองจุดที่มี coordinates ใน F_1 คือ $\mathbf{p}$ และ $\mathbf{q}$ โดย coordinates ของสองจุดนี้ใน F_0 เป็น $\mathbf{P}$ และ $\mathbf{Q}$ ตามลำดับ เราได้ว่า

$$
\begin{aligned}
|\mathbf{P}-\mathbf{Q}| &= |([A]\mathbf{p}+\mathbf{d})-([A]\mathbf{q}+\mathbf{d})| \\
&= |[A](\mathbf{p}-\mathbf{q})| \\
&= \sqrt{(\mathbf{p}-\mathbf{q})^T[A^T][A](\mathbf{p}-\mathbf{q})}
\end{aligned}
$$

เพราะ $|\mathbf{p}\text{-}\mathbf{q}|$ ต้องเท่ากับ $|\mathbf{P}\text{-}\mathbf{Q}|$ นั่นคือ

$$[A^T][A]=[I]$$

The constraint $[A^T][A]=[I]$ ensures that $\mathbf{X}=[A]\mathbf{x}+\mathbf{d}$ is a rigid transformation.

Premultiply and postmultiply both sides of the constraint by [A] and [A^{-1}], we obtain:

$$[A][A^T][A][A^{-1}]=[A][I][A^{-1}]$$

That is, $[A][A^T]=[I]$

Matrix [A] such that: $[A^T][A] = [I]$

or $[A][A^T] = [I]$

is called ***orthogonal*** matrix.

Let $[A] = [\mathbf{a_1}\ \mathbf{a_2} \ldots \mathbf{a_m}]$

This means $\mathbf{a_i} \bullet \mathbf{a_j} = \begin{cases} 1 & \text{i=j} \\ 0 & \text{i} \neq \text{j} \end{cases}$

Using determinant identity det(A)=det(A^T), we obtain:

$$\det([A^T][A]) = \det([A])^2 = \det([I]) = 1$$

This implies: $\det([A]) = \pm 1$

[A] corresponds to a rotation only when det([A])=1. When det([A])= -1, the matrix corresponds to a reflection.

Coordinate transformation can also be viewed as a displacement.

As we can see that the translation of the sum of two vectors is not the sum of the translation of each vector separately, displacement is not a direct linear transformation.

So displacement in n dimensional space **cannot** be represented by $n \times n$ matrix transformation.

To write the transformation in a matrix form, we use homogeneous transformation.

Homogeneous Transformation

Key: $\mathbb{R}^n$ is embeded as a hyperplane in $\mathbb{R}^{n+1}$

Displacement can then be represented by a matrix

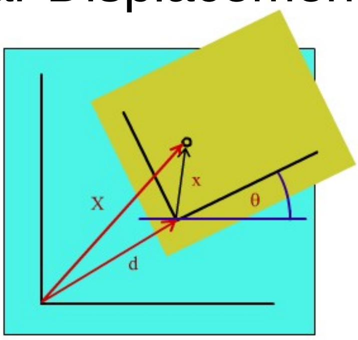
$$\begin{Bmatrix} \mathbf{X} \\ \mathbf{1} \end{Bmatrix} = \begin{bmatrix} A & \mathbf{d} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \mathbf{1} \end{Bmatrix}$$

Homogeneous transforms form a matrix group.

Let $[T_1]$ and $[T_2]$ be matrix of homogeneous transforms, we can show that $[T_1][T_2]$ is also a matrix of homogeneous transform.

Likewise, inverse transform can be obtained from the inverse of the matrix of the transform.

Planar Displacements



$$\mathbf{X} = [A]\mathbf{x} + \mathbf{d}$$ where

$$[A] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \mathbf{d} = \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

We can see [A][A^T] =

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} =$$

$$\begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [I]$$

That is [*A*] is an orthogonal matrix and because det([*A*])=1, we know that [*A*] is a rotation.

Now consider an orthogonal 2 x 2 matrix which is not a rotation:

$$[X] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

What does [X] do?

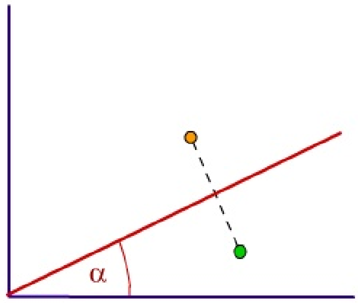
$$[X] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

It reflects coordinates of points in the plane through the line x = 0. For example,

$$[X]\begin{Bmatrix} 5 \\ 6 \end{Bmatrix} = \begin{Bmatrix} -5 \\ 6 \end{Bmatrix}$$

A reflection through a line at an angle α about the origin is given by:

$$[S] = [A][X][A^T] = \begin{bmatrix} -\cos 2\alpha & -\sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{bmatrix}$$



What if several reflections are applied?

The determinant of a product of n reflections is $(-1)^n$, therefore if n is an even number, the product is a rotation, not a reflection.

For example, let [*S*] and [*T*] be reflections through lines at the angles α and β about the origin respectively, then the product [*S*][*T*] is the rotation:

$$[S][T] = \begin{bmatrix} \cos 2(\alpha-\beta) & -\sin 2(\alpha-\beta) \\ \sin 2(\alpha-\beta) & \cos 2(\alpha-\beta) \end{bmatrix}$$

Pole of a Planar Displacement

For a general planar displacement, there is a point that does not move. This point is called the pole of the displacement.

Let D=(A,**d**) be the displacement, the its pole **p** satisfies the equation D**p**=**p,** or

$$\mathbf{p} = [A]\mathbf{p} + \mathbf{d}$$

Solving for **p** yields

$$\mathbf{p} = -[A - I]^{-1}\mathbf{d}$$, or

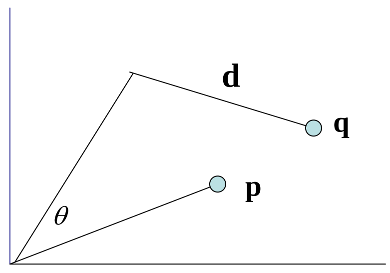
$$p_1 = \frac{d_1 \sin(\theta/2) - d_2 \cos(\theta/2)}{2\sin(\theta/2)}$$

$$p_2 = \frac{d_1 \cos(\theta/2) - d_2 \sin(\theta/2)}{2\sin(\theta/2)}$$

The only case for which this does not have a solution is when $\theta=0$ (pure translation). In this case the coordinates of the pole move to infinity along the line perpendicular to **d**.

Any general planar displacement can be written in the form of a rotation around a pole.

Pole = Center of Rotation



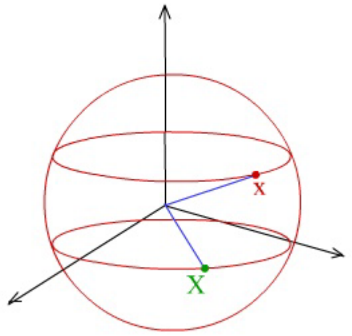
$$T(\theta,\mathbf{d}) = \begin{bmatrix} \cos\theta & -\sin\theta & d_x \\ \sin\theta & \cos\theta & d_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{q} = T(\theta,\mathbf{d})\mathbf{p}$$

3D Rotation

A three-dimensional rotation can be represented by the transformation equation

$$\mathbf{X} = [A]\mathbf{x}$$



Because **X** and **x** must be on a sphere, we must have

$$\mathbf{X}^T\mathbf{X} = \mathbf{x}^T\mathbf{x}$$

This requires $[A]^T[A] = [I]$

Which means [A] is an orthogonal matrix. Rotations are orthogonal matrices with determinant of 1. They form the matrix group called SO(3).

Orthogonal matrices with determinant –1 are reflection. An example is

$$[X] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Which reflects coordinates through the plane X=0. Like in 2D case, a general reflection is obtained as the product [A][X][A^T] where [A] is a rotation matrix.

Fundamental Rotation (Euler Angles)

Yaw: counterclockwise rotation α about z axis

$$R_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

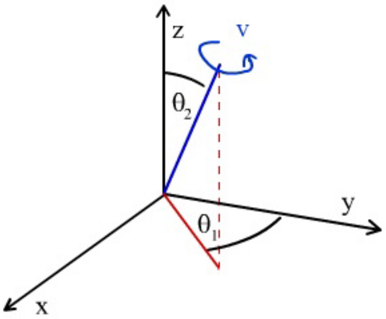
Pitch: counterclockwise rotation β about y axis

$$R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

Roll: counterclockwise rotation γ about x axis

$$R_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

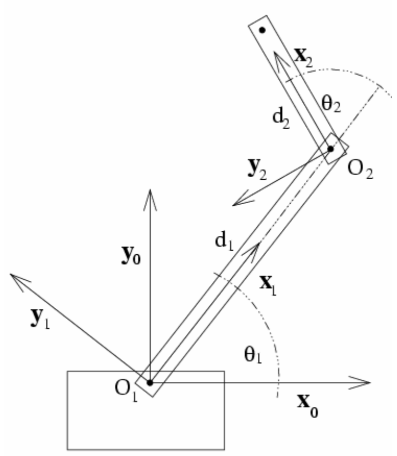
Rotation around an arbitrary axis



1. Rotate **v** to make it coincide with **z**
2. Rotate about **z**
3. Inverse step 1

$$[R_x(\theta_2)R_z(\theta_1)]^{-1}R_z(\theta)[R_x(\theta_2)R_z(\theta_1)]$$

Forward Kinematics



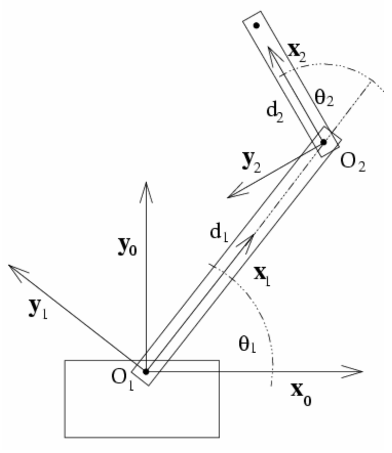
กำหนดให้

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{D}(a,b) = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

Frame F_i มี $\mathbf{x}_\mathrm{i}$ และ $\mathbf{y}_\mathrm{i}$ เป็นแกน
และ O_i เป็นจุดกำเนิด

Forward Kinematics

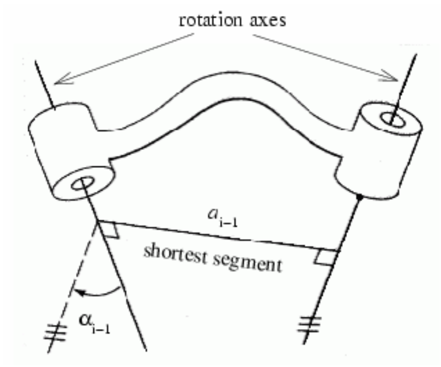


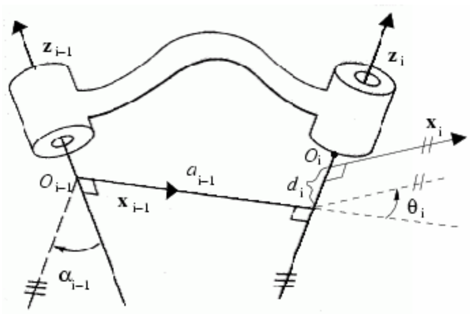
พิกัดของ F_2 เขียนในรูปพิกัดของ F_1 โดย

- เขียนพิกัดของ F_1 ในรูปพิกัดของ F_0 ทำได้ด้วยการ premultiply พิกัดใน F_1 ด้วย $\mathbf{R}(\theta_1)$
- เขียนพิกัดของ F_2 ในรูปพิกัดของ F_1 ทำได้ด้วยการ premultiply พิกัดใน F_2 ด้วย $\mathbf{D}(d_1,0)\mathbf{R}(\theta_2)$

ดังนั้นจุดที่มีพิกัด $\mathbf{p}$ ใน F_2 มีพิกัดใน F_0 คือ

$\mathbf{R}(\theta_1)\mathbf{D}(d_1,0)\mathbf{R}(\theta_2)\mathbf{p}$



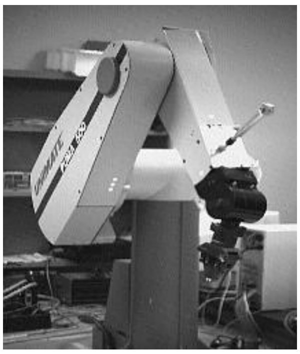


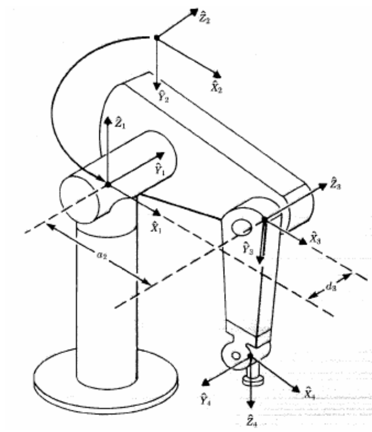
$$\mathbf{T}_{i-1,i} = \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & \cos\alpha_{i-1} & -\sin\alpha_{i-1} & 0 \\ 0 & \sin\alpha_{i-1} & \cos\alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}_{i-1,i} = \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & \cos\alpha_{i-1} & -\sin\alpha_{i-1} & 0 \\ 0 & \sin\alpha_{i-1} & \cos\alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

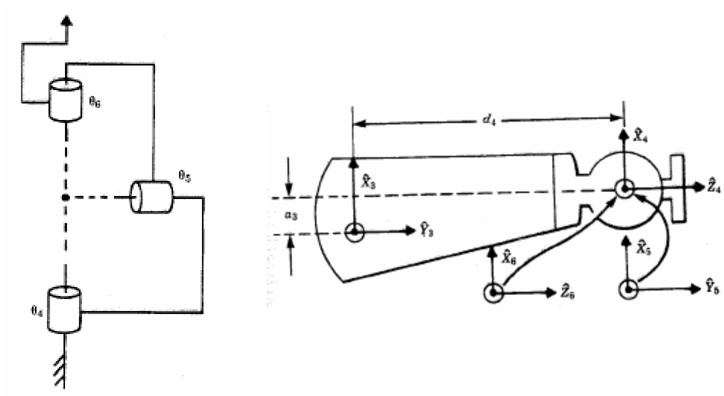
$$= \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & a_{i-1} \\ \sin\theta_i \cos\alpha_{i-1} & \cos\theta_i \cos\alpha_{i-1} & -\sin\alpha_{i-1} & -d_i \sin\alpha_{i-1} \\ \sin\theta_i \sin\alpha_{i-1} & \cos\theta_i \sin\alpha_{i-1} & \cos\alpha_{i-1} & d_i \cos\alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

PUMA 560





PUMA 560



PUMA 560

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	$-\pi/2$	0	0	θ_2
3	0	a_2	d_3	θ_3
4	$-\pi/2$	a_3	d_4	θ_4
5	$\pi/2$	0	0	θ_5
6	$-\pi/2$	0	0	θ_6

$$\mathbf{T}_{0,1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{T}_{1,2} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\theta_2 & -\cos\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}_{2,3} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & a_2 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{T}_{3,4} = \begin{bmatrix} \cos\theta_4 & -\sin\theta_4 & 0 & a_3 \\ 0 & 0 & 1 & d_4 \\ -\sin\theta_4 & -\cos\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}_{4,5} = \begin{bmatrix} \cos\theta_5 & -\sin\theta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sin\theta_5 & \cos\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{T}_{5,6} = \begin{bmatrix} \cos\theta_6 & -\sin\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\theta_6 & -\cos\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}_{0,6} = \mathbf{T}_{0,1}\mathbf{T}_{1,2}\mathbf{T}_{2,3}\mathbf{T}_{3.4}\mathbf{T}_{4,5}\mathbf{T}_{5,6} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$r_{11} = \cos\theta_1\left[\cos(\theta_2+\theta_3)(\cos\theta_4\cos\theta_5\cos\theta_6 - \sin\theta_4\sin\theta_6) - \sin(\theta_2+\theta_3)\sin\theta_5\cos\theta_6\right] + \sin\theta_1\left[\sin\theta_4\cos\theta_5\cos\theta_6 + \cos\theta_4\sin\theta_6\right]$$

$$r_{21} = \sin\theta_1\left[\cos(\theta_2+\theta_3)(\cos\theta_4\cos\theta_5\cos\theta_6 - \sin\theta_4\sin\theta_6) - \sin(\theta_2+\theta_3)\sin\theta_5\cos\theta_6\right] - \cos\theta_1\left[\sin\theta_4\cos\theta_5\cos\theta_6 + \cos\theta_4\sin\theta_6\right]$$

$$r_{31} = -\sin(\theta_2+\theta_3)(\cos\theta_4\cos\theta_5\cos\theta_6 - \sin\theta_4\sin\theta_6) - \cos(\theta_2+\theta_3)\sin\theta_5\cos\theta_6$$

$$r_{12}=\cos\theta_1\left[\cos(\theta_2+\theta_3)(-\cos\theta_4\cos\theta_5\cos\theta_6-\sin\theta_4\cos\theta_6)+\sin(\theta_2+\theta_3)\sin\theta_5\sin\theta_6\right]+$$
$$\sin\theta_1\left[\cos\theta_4\cos\theta_6-\sin\theta_4\cos\theta_5\sin\theta_6\right]$$

$$r_{22}=\sin\theta_1\left[\cos(\theta_2+\theta_3)(-\cos\theta_4\cos\theta_5\cos\theta_6-\sin\theta_4\cos\theta_6)+\sin(\theta_2+\theta_3)\sin\theta_5\sin\theta_6\right]-$$
$$\cos\theta_1\left[\cos\theta_4\cos\theta_6-\sin\theta_4\cos\theta_5\sin\theta_6\right]$$

$$r_{32}=-\sin(\theta_2+\theta_3)(-\cos\theta_4\cos\theta_5\sin\theta_6-\sin\theta_4\cos\theta_6)+\cos(\theta_2+\theta_3)\sin\theta_5\sin\theta_6$$

$$r_{13}=-\cos\theta_1\left(\cos(\theta_2+\theta_3)\cos\theta_4\sin\theta_5+\sin(\theta_2+\theta_3)\cos\theta_5\right)-\sin\theta_1\sin\theta_4\sin\theta_5$$

$$r_{23}=-\sin\theta_1\left(\cos(\theta_2+\theta_3)\cos\theta_4\sin\theta_5+\sin(\theta_2+\theta_3)\cos\theta_5\right)-\cos\theta_1\sin\theta_4\sin\theta_5$$

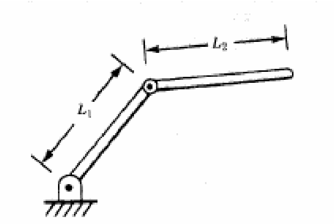
$$r_{33}=\sin(\theta_2+\theta_3)\cos\theta_4\sin\theta_5-\cos(\theta_2+\theta_3)\cos\theta_5$$

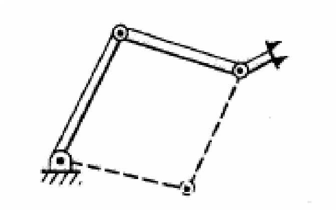
$$p_x=\cos\theta_1\left[a_2\cos\theta_2+a_3\cos(\theta_2+\theta_3)-d_4\sin(\theta_2+\theta_3)\right]-d_3\sin\theta_1$$

$$p_y=\sin\theta_1\left[a_2\cos\theta_2+a_3\cos(\theta_2+\theta_3)-d_4\sin(\theta_2+\theta_3)\right]+d_3\cos\theta_1$$

$$p_z=-a_3\sin(\theta_2+\theta_3)-a_2\sin\theta_2-d_4\cos(\theta_2+\theta_3)$$

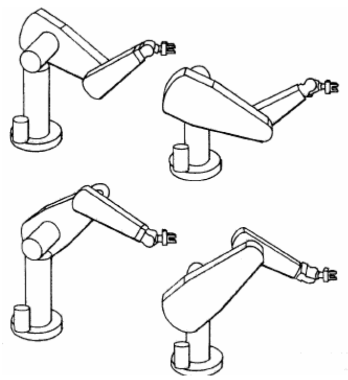
Inverse Kinematics





คำนวณพารามิเตอร์ของข้อต่อที่ทำให้ปลายแขนอยู่ในพิกัดที่กำหนด

Inverse Kinematics

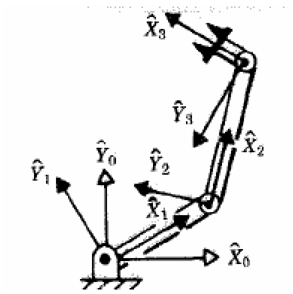


Solution:

(1) Not necessarily unique

(2) May not exist

Inverse Kinematics: Example



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	l_1	0	θ_2
3	0	l_2	0	θ_3

Inverse Kinematics: Example

$$\mathbf{T}_{0,3} = \begin{bmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & 0 & l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) & 0 & l_1\sin\theta_1 + l_2\sin(\theta_1+\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}_{0,3} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 & x \\ \sin\phi & \cos\phi & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse Kinematics: Example

เพราะ $x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$ และ

$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$

จึงได้ว่า $x^2 + y^2 = l_1^2 + l_2^2 + 2l_1 l_2 \cos\theta_2$

และ $\cos\theta_2 = \dfrac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1 l_2}$

$\sin\theta_2 = \pm\sqrt{1 - \cos^2\theta_2}$

ซึ่งก็คือ $\theta_2 = \operatorname{atan2}(\sin\theta_2, \cos\theta_2)$

Inverse Kinematics: Example

จัดรูปใหม่ของ $x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$ และ $y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$

ได้ $x = k_1 \cos\theta_1 - k_2 \sin\theta_1$ และ $y = k_1 \sin\theta_1 + k_2 \cos\theta_1$

โดย $k_1 = l_1 + l_2 \cos\theta_2$ และ $k_2 = l_2 \sin\theta_2$

เมื่อกำหนดให้ $r = +\sqrt{k_1^2 + k_2^2}$ และ $\gamma = \operatorname{atan2}(k_2 / r, k_1 / r)$

ได้ $k_1 = r\cos\gamma$ และ $k_2 = r\sin\gamma$

จึงได้ $x / r = \cos\gamma\cos\theta_1 - \sin\gamma\sin\theta_1 = \cos(\gamma + \theta_1)$

และ $y / r = \cos\gamma\sin\theta_1 + \sin\gamma\cos\theta_1 = \sin(\gamma + \theta_1)$

Inverse Kinematics: Example

ทำให้ได้ $\gamma + \theta_1 = \text{atan2}(y/r, x/r)$

และ $\theta_1 = \text{atan2}(y/r, x/r) - \text{atan2}(k_2/r, k_1/r)$

จาก $\cos\phi = \cos(\theta_1 + \theta_2 + \theta_3)$

และ $\sin\phi = \sin(\theta_1 + \theta_2 + \theta_3)$

เมื่อรู้ θ_1, θ_2 จึงหา θ_3 ได้