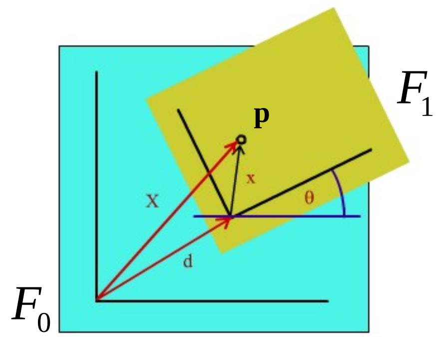
Displacement

Coordinate Transformation



จุด p มี coordinates ใน F_1 คือ x

สมมุติว่า coordinates ของมันใน F_0 เขียนได้เป็น

$$\mathbf{X} = [A]\mathbf{x} + \mathbf{d}$$

พิจารณาสองจุดที่มี coordinates ใน F_1 คือ **p** และ **q** โดย coordinates ของสองจุดนี้ใน F_0 เป็น **P** และ **Q** ตามลำดับ เราได้ว่า

$$
\begin{aligned}
|\mathbf{P}-\mathbf{Q}| &= |([A]\mathbf{p}+\mathbf{d})-([A]\mathbf{q}+\mathbf{d})| \\
&= |[A](\mathbf{p}-\mathbf{q})| \\
&= \sqrt{(\mathbf{p}-\mathbf{q})^T[A^T][A](\mathbf{p}-\mathbf{q})}
\end{aligned}
$$

เพราะ |**p-q**| ต้องเท่ากับ |**P-Q**| นั่นคือ

$$[A^T][A]=[I]$$

The constraint $[A^T][A]=[I]$ ensures that $\mathbf{X}=[A]\mathbf{x}+\mathbf{d}$ is a rigid transformation.

Premultiply and postmultiply both sides of the constraint by [A] and [A^{-1}], we obtain:

$$[A][A^T][A][A^{-1}]=[A][I][A^{-1}]$$

That is, $[A][A^T]=[I]$

Matrix [A] such that: $[A^T][A] = [I]$

or $[A][A^T] = [I]$

is called ***orthogonal*** matrix.

Let $[A] = [\mathbf{a_1}\ \mathbf{a_2} \ldots \mathbf{a_m}]$

This means $\mathbf{a}_i \cdot \mathbf{a}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

Using determinant identity det(A)=det(A^T), we obtain:

$$\det([A^T][A]) = \det([A])^2 = \det([I]) = 1$$

This implies: $\det([A]) = \pm 1$

[A] corresponds to a rotation only when det([A])=1. When det([A])= -1, the matrix corresponds to a reflection.

Coordinate transformation can also be viewed as a displacement.

As we can see that the translation of the sum of two vectors is not the sum of the translation of each vector separately, displacement is not a direct linear transformation. For example, let T(**v**) be the translation **v**+**d**, it is clear that T(**v**+**w**) != T(**v**) + T(**w**)

So displacement in n dimensional space **cannot** be represented by n x n matrix transformation.

To write the transformation in a matrix form, we use homogeneous transformation.

Homogeneous Transformation

Key: $\Re^n$ is embeded as a hyperplane in $\Re^{n+1}$

Displacement can then be represented by a matrix

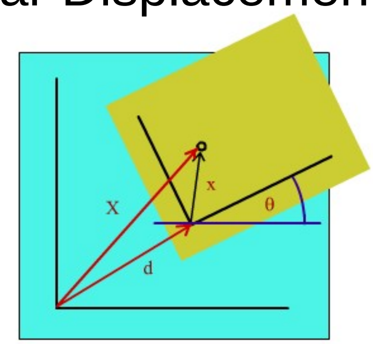
$$\begin{Bmatrix} \mathbf{X} \\ \mathbf{1} \end{Bmatrix} = \begin{bmatrix} A & \mathbf{d} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \mathbf{1} \end{Bmatrix}$$

Homogeneous transforms form a matrix group.

Let $[T_1]$ and $[T_2]$ be matrix of homogeneous transforms, we can show that $[T_1][T_2]$ is also a matrix of homogeneous transform.

Likewise, inverse transform can be obtained from the inverse of the matrix of the transform.

Planar Displacements



$$\mathbf{X} = [A]\mathbf{x} + \mathbf{d}$$ where

$$[A] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \mathbf{d} = \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

We can see [A][A^T] =

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} =$$

$$\begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [I]$$

That is [*A*] is an orthogonal matrix and because det([*A*])=1, we know that [*A*] is a rotation.

Now consider an orthogonal 2 x 2 matrix which is not a rotation:

$$[X] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

What does [X] do?

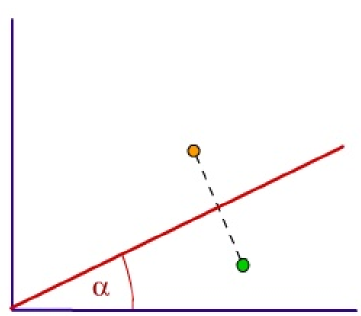
$$[X] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

It reflects coordinates of points in the plane through the line $x = 0$. For example,

$$[X]\begin{Bmatrix} 5 \\ 6 \end{Bmatrix} = \begin{Bmatrix} -5 \\ 6 \end{Bmatrix}$$

A reflection through a line at an angle α about the origin is given by:

$$[S] = [A][X][A^T] = \begin{bmatrix} -\cos 2\alpha & -\sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{bmatrix}$$



What if several reflections are applied?

The determinant of a product of n reflections is $(-1)^n$, therefore if n is an even number, the product is a rotation, not a reflection.

For example, let [S] and [T] be reflections through lines at the angles α and β about the origin respectively, then the product [S][T] is the rotation:

$$[S][T] = \begin{bmatrix} \cos 2(\alpha - \beta) & -\sin 2(\alpha - \beta) \\ \sin 2(\alpha - \beta) & \cos 2(\alpha - \beta) \end{bmatrix}$$

Pole of a Planar Displacement

For a general planar displacement, there is a point that does not move. This point is called the pole of the displacement.

Let D=(A,**d**) be the displacement, the its pole **p** satisfies the equation D**p**=**p,** or

$$\mathbf{p} = [A]\mathbf{p} + \mathbf{d}$$

Solving for **p** yields

$$\mathbf{p} = -[A - I]^{-1}\mathbf{d}$$, or

$$p_1 = \frac{d_1 \sin(\theta/2) - d_2 \cos(\theta/2)}{2\sin(\theta/2)}$$

$$p_2 = \frac{d_1 \cos(\theta/2) - d_2 \sin(\theta/2)}{2\sin(\theta/2)}$$

The only case for which this does not have a solution is when $\theta=0$ (pure translation). In this case the coordinates of the pole move to infinity along the line perpendicular to **d**.

Any general planar displacement can be written in the form of a rotation around a pole.

Pole = Center of Rotation

Poles using Homogeneous Transform

$$\begin{Bmatrix} X \\ Y \\ 1 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & d_1 \\ \sin\theta & \cos\theta & d_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ 1 \end{Bmatrix}.$$

$$\begin{Bmatrix} \mathbf{p} \\ 1 \end{Bmatrix} = \begin{bmatrix} A & \mathbf{d} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \mathbf{p} \\ 1 \end{Bmatrix},$$

Poles using Homogeneous Transform

Recall the Eigen form: **Ax** = λ**x**

$$\det \begin{bmatrix} \cos\theta - \lambda & -\sin\theta & d_1 \\ \sin\theta & \cos\theta - \lambda & d_2 \\ 0 & 0 & 1-\lambda \end{bmatrix} = 0,$$

$$(1-\lambda)(\lambda^2 - 2\lambda\cos\theta + 1) = 0$$

In homogeneous coordinates, **x** is the same as λ**x**

Poles using Homogeneous Transform

The 3×3 matrix $[A, \mathbf{d}]$ obviously has $\lambda = 1$ as an eigenvalue. The remaining two are $\lambda = \exp(i\theta)$ and $\lambda^* = \exp(-i\theta)$ which are also the eigenvalues of the rotation matrix $[A]$. Let $\mathbf{x} = (x_1, x_2, 0)$ and $\mathbf{x}^* = (x_1^*, x_2^*, 0)$ be the eigenvectors associated with this latter pair of eigenvalues. A pair of real orthogonal vectors, $\mathbf{c}_1$ and $\mathbf{c}_2$, can be constructed from $\mathbf{x}$ and $\mathbf{x}^*$ by the formulae

$$\mathbf{c}_1 = (\mathbf{x} + \mathbf{x}^*)/2$$
$$\mathbf{c}_2 = i(\mathbf{x} - \mathbf{x}^*)/2$$

Remember **x** is homogeneous!

Poles using Homogeneous Transform

Starting from a taylor series expansion of the exponential function

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$
$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \ldots$$
$$= (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots) + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots)$$
$$= \cos x + i \sin x$$

Poles using Homogeneous Transform

The fixed frame coordinates $\mathbf{C}_1$ and $\mathbf{C}_2$ of these points are obtained by the computation

$$\begin{aligned}\mathbf{C}_1 = [A]\mathbf{c}_1 &= (\exp(i\theta)\mathbf{x} + \exp(-i\theta)\mathbf{x}^*)/2,\\ \mathbf{C}_2 = [A]\mathbf{c}_2 &= i(\exp(i\theta)\mathbf{x} - \exp(-i\theta)\mathbf{x}^*)/2,\end{aligned}$$

or

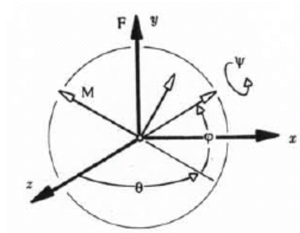
$$\begin{aligned}\mathbf{C}_1 &= \cos\theta\mathbf{c}_1 + \sin\theta\mathbf{c}_2,\\ \mathbf{C}_2 &= -\sin\theta\mathbf{c}_1 + \cos\theta\mathbf{c}_2.\end{aligned}$$

Poles using Homogeneous Transform

The eigenvector associated with $\lambda = 1$ satisfies the equation

$$\begin{bmatrix} A - I & \mathbf{d} \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{p} \\ 1 \end{Bmatrix} = 0.$$

Spherical Displacement



$$\mathbf{X} = [A]\mathbf{x},$$

Spherical Displacement

$$\mathbf{X}^T\mathbf{X} = \mathbf{x}^T[A^T][A]\mathbf{x} = \mathbf{x}^T\mathbf{x},$$

$$[A^T][A] = [I],$$

which means $[A]$ is an orthogonal matrix. Rotations are orthogonal matrices with determinant equal to 1. They form the matrix group denoted $SO(3)$.

Spherical Displacement

Orthogonal matrices with determinant equal to -1 are called reflections. A typical example is

$$[X] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection about the plane $x = 0$

Fundamental Rotation (Euler Angles)

Yaw: counterclockwise rotation α about z axis

$$R_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

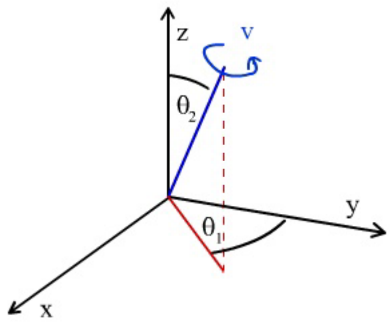
Pitch: counterclockwise rotation β about y axis

$$R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

Roll: counterclockwise rotation γ about x axis

$$R_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

Rotation around an arbitrary axis



1. Rotate **v** to make it coincide with **z**
2. Rotate about **z**
3. Inverse step 1

$$[R_x(\theta_2)R_z(\theta_1)]^{-1}R_z(\theta)[R_x(\theta_2)R_z(\theta_1)]$$

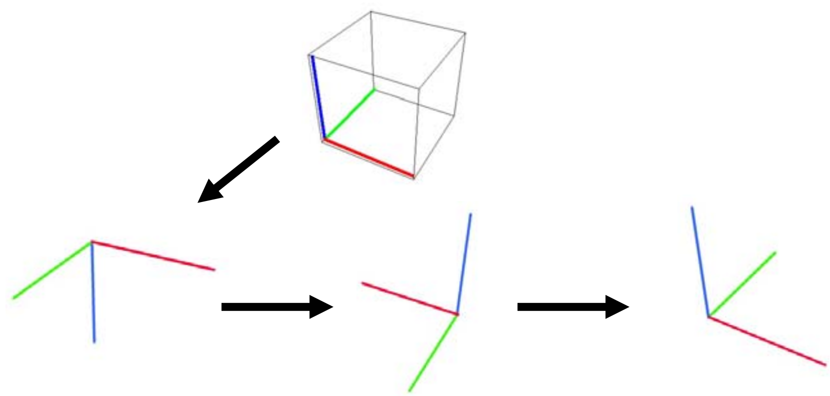
Euler Angles

Drawback

It is possible that $R_z(\alpha)R_y(\beta)R_x(\gamma) = I$
when $\alpha, \beta, \gamma \neq 0$

A single rotation can be represented by many combinations of α, β, γ

Euler Angles



Euler Angles

Examples

Consider $R = R_z(\alpha)R_y(\beta)R_x(\gamma)$

where $\alpha = \beta = \gamma = \pi$

We thus have:

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Euler Angles

Drawback

It is possible that $R_z(\alpha)R_y(\beta)R_x(\gamma) = I$
when $\alpha, \beta, \gamma \neq 0$

A single rotation can be represented by many combinations of α, β, γ

Gimbal Lock

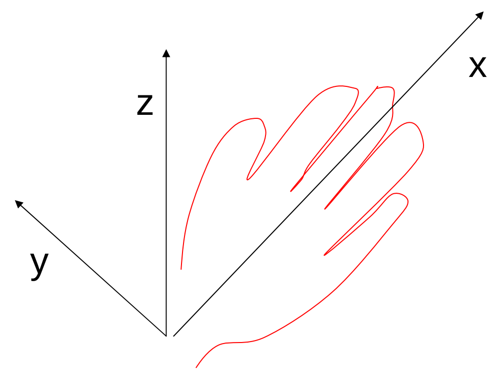
Problem with Euler angle

Consider Y-P-R with 90 degree pitch

Two parameters from yaw and roll can only give one degree of freedom to the rotation.

Demo with the hand!

Gimbal Lock



A solution: quaternion!

Eigenvector of an Orthogonal Matrix

$$[A]\mathbf{x} = \lambda\mathbf{x}$$

$$[A]\mathbf{x} = \mathbf{x}$$

$\mathbf{x}$ is now in ordinary cartesian coordinates

det(**A**-λ**I**) = 0

$$-\lambda^3 + \lambda^2(a_{11} + a_{22} + a_{33}) - \lambda(M_{11} + M_{22} + M_{33}) + \det(A) = 0$$

$M_{i,j}$ denotes minor obtained from eliminating row i and column j.

Eigenvector of an Orthogonal Matrix

$$-\lambda^3 + \lambda^2(a_{11} + a_{22} + a_{33}) - \lambda(M_{11} + M_{22} + M_{33}) + \det(A) = 0$$

When **A** is an orthogonal matrix, $M_{ii} = a_{ii}$

Why?

$$|A| = \sum_{i=1}^{k} a_{ij} C_{ij},$$

$$C_{ij} \equiv (-1)^{i+j} M_{ij}.$$

Eigenvector of an Orthogonal Matrix

For a 3x3 orthogonal matrix A, |A|=1 and $a_{i,1}^2 + a_{i,2}^2 + a_{i,3}^2 = 1$ for *I* = 1,2 or 3.

Compare with the determinant formula from the last page, we obtain:

$$C_{i,j} = a_{i,j}$$

or

$$M_{i,j} = a_{i,j} / (-1)^{i+j}$$

Eigenvector of an Orthogonal Matrix

When **A** is an orthogonal matrix, $M_{ii} = a_{ii,}$ resulting in

$$\lambda^3 - \lambda^2(a_{11} + a_{22} + a_{33}) + \lambda(a_{11} + a_{22} + a_{33}) - 1 = 0.$$

We see immediately that $\lambda = 1$ is a root, so the characteristic polynomial can be factored to obtain

$$(\lambda - 1)(\lambda^2 - \lambda(a_{11} + a_{22} + a_{33} + 1) + 1) = 0.$$

The remaining roots are: $\lambda = e^{i\phi}$ and $\lambda^* = e^{-i\phi}$

where $\cos\phi = (a_{11} + a_{22} + a_{33} - 1)/2$

Eigenvector of an Orthogonal Matrix

Let **b** be the Eigenvector associated with Eigenvalue λ=1

We have A **b** = **b**. That is, *t* **b** is fixed under the rotation given by A

The other two Eigenvectors (say **x** and **x***) span the plane orthogonal to **b**

We can construct real orthogonal vectors in this plane by

$$\mathbf{c}_1 = (\mathbf{x} + \mathbf{x}^*)/2$$
$$\mathbf{c}_2 = i(\mathbf{x} - \mathbf{x}^*)/2$$

Eigenvector of an Orthogonal Matrix

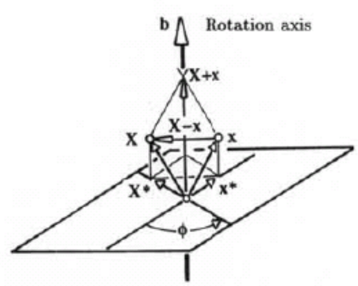
$$\mathbf{C}_1 = [A]\mathbf{c}_1 = (\exp(i\phi)\mathbf{x} + \exp(-i\phi)\mathbf{x}^*)/2,$$
$$\mathbf{C}_2 = [A]\mathbf{c}_2 = i(\exp(i\phi)\mathbf{x} - \exp(-i\phi)\mathbf{x}^*))/2,$$

or

$$\mathbf{C}_1 = \cos\phi\mathbf{c}_1 + \sin\phi\mathbf{c}_2,$$
$$\mathbf{C}_2 = -\sin\phi\mathbf{c}_1 + \cos\phi\mathbf{c}_2.$$

Still in the plane spanned by $\mathbf{c}_1$ and $\mathbf{c}_2$

Cayley's Formula



$$(\mathbf{X} - \mathbf{x})^T(\mathbf{X} + \mathbf{x}) = 0.$$

$$\mathbf{X} - \mathbf{x} = [A - I]\mathbf{x},$$

$$\mathbf{X} + \mathbf{x} = [A + I]\mathbf{x}$$

$$\mathbf{X} - \mathbf{x} = [A - I][A + I]^{-1}(\mathbf{X} + \mathbf{x}).$$

$$[B] = [A - I][A + I]^{-1}$$

$[B]\mathbf{y}$ is orthogonal to $\mathbf{y}$

$$\mathbf{y}^T[B]\mathbf{y} = \sum(b_{ij} + b_{ji})y_i y_j = 0.$$

Cayley's Formula

For $\mathbf{y}^T[B]\mathbf{y} = \sum(b_{ij} + b_{ji})y_i y_j = 0.$ to be true:

$[B] = -[B^T]$, which is termed *skew-symmetry*.

We solve $[B] = [A - I][A + I]^{-1}$ for $[A]$ to obtain *Cayley's formula* for an orthogonal matrix

$$[A] = [I - B]^{-1}[I + B].$$

[*I-B*] cannot be singular because it is skew-symmetric, which means it only has imaginary Eigenvalues (if a matrix is singular, it has 1 as an eigenvalue). Detail in the next few slides.

Cayley's Formula

The matrices $[I + B]$ and $[I - B]$ commute, since

$$[I + B][I - B] = [I - B][I + B] = [I - B^2].$$

Pre-multiplying and post-multiplying this equation by $[I - B]^{-1}$ results in the fact that $[I + B]$ and $[I - B]^{-1}$ also commute. Thus Cayley's formula has the equivalent form

$$[A] = [I + B][I - B]^{-1}.$$

Skew-Symmetric Matrices

$$[B] = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}.$$

$$\mathbf{b} = (b_1, b_2, b_3)$$

$$[B]\mathbf{y} = \mathbf{b} \times \mathbf{y}$$

Vector **b** here is the eigenvector of an orthogonal matrix *A* associated with eigenvalue 1

With λ=1, $[A-I]\mathbf{x} = 0$.

Replace $[A]$ by Cayley's formula, multiply by $[I - B]$, and simplify the result to obtain

$[B]\mathbf{x} = 0$. Since $[B]\mathbf{x} = \mathbf{b} \times \mathbf{x}$, the solution is $\mathbf{x} = \mathbf{b}$

Skew-Symmetric Matrices

The eigenvalues λ of a general 3×3 skew-symmetric matrix $[W]$ satisfy the equation:

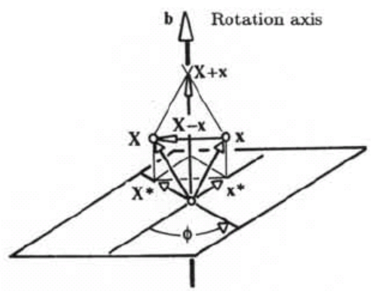
$$\det \begin{bmatrix} -\lambda & -w_3 & w_2 \\ w_3 & -\lambda & -w_1 \\ -w_2 & w_1 & -\lambda \end{bmatrix} = 0,$$

which simplifies to the characteristic equation

$$\lambda^3 + (w_1^2 + w_2^2 + w_3^2)\lambda = 0.$$

This means that λ can either be zero or imaginary.

Rodrigue’s Equation



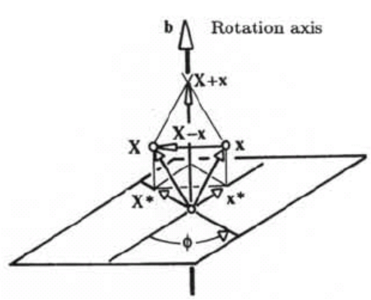
$$\mathbf{X} - \mathbf{x} = [B](\mathbf{X} + \mathbf{x})$$

$$(\mathbf{X} - \mathbf{x}) = \mathbf{b} \times (\mathbf{X} + \mathbf{x})$$

Let **x*** and **X*** be the normal projection of **x** and **X** on the plane orthogonal to the rotation axis **b**

$$|\mathbf{X}^* - \mathbf{x}^*| = |\mathbf{b}||\mathbf{X}^* + \mathbf{x}^*|$$

Rodrigue’s Equation



$$|\mathbf{X}^* - \mathbf{x}^*| = |\mathbf{b}||\mathbf{X}^* + \mathbf{x}^*|$$

$$\tan(\frac{\phi}{2}) = \frac{|\mathbf{X}^* - \mathbf{x}^*|}{|\mathbf{X}^* + \mathbf{x}^*|}.$$

$$|\mathbf{b}| = \tan(\phi/2)$$

$$b_1 = \tan(\phi/2)s_x, b_2 = \tan(\phi/2)s_y, b_3 = \tan(\phi/2)s_z$$

where $\mathbf{s} = (s_x, s_y, s_z)$ is the unit vector along $\mathbf{b}$

These constants are the Rodrigue’s parameters.

Euler Parameters

$[B] = \tan(\phi/2)[S]$

Plug this into cayley formula $[A] = [I - B]^{-1}[I + B]$

$$[A] = [\cos(\frac{\phi}{2})I - \sin(\frac{\phi}{2})S]^{-1}[\cos(\frac{\phi}{2})I + \sin(\frac{\phi}{2})S]$$

The constants in $[C] = [\cos(\phi/2)I + \sin(\phi/2)S]$ are the *Euler parameters* of $[A]$, given by

$$c_0 = \cos(\phi/2), c_1 = \sin(\phi/2)s_x, c_2 = \sin(\phi/2)s_y, c_3 = \sin(\phi/2)s_z$$

Euler Parameters

Plugging this

$$[\cos(\frac{\phi}{2})I - \sin(\frac{\phi}{2})S]^{-1} = \cos(\frac{\phi}{2})[I] + \sin(\frac{\phi}{2})[S] + \frac{\sin^2(\phi/2)}{\cos(\phi/2)}[I + S^2]$$

into

$$[A] = [\cos(\frac{\phi}{2})I - \sin(\frac{\phi}{2})S]^{-1}[\cos(\frac{\phi}{2})I + \sin(\frac{\phi}{2})S]$$

and multiply by [C], we obtain:

$$[A] = I + 2\sin(\frac{\phi}{2})\cos(\frac{\phi}{2})[S] + 2\sin^2(\frac{\phi}{2})[S^2]$$

(this uses identity from unit skew symmetry: $[S^3]+[S]=0$)

Euler Parameters

$$[A] = I + 2\sin(\frac{\phi}{2})\cos(\frac{\phi}{2})[S] + 2\sin^2(\frac{\phi}{2})[S^2]$$

$$[A] = [I] + \sin\phi[S] + (1-\cos\phi)[S^2]$$

Using skew symmetry of [*S*] and symmetry of [*I*], we obtain:

$$2\sin\phi[S] = [A - A^T]$$

Rodrigue's Formula

$$\dot{q}(t) = \omega \times q(t) = \hat{\omega} q(t)$$
$$q(t) = e^{\hat{\omega}t} q(0)$$
$$e^{\hat{\omega}t} = I + \hat{\omega}t + \frac{(\hat{\omega}t)^2}{2!} + \frac{(\hat{\omega}t)^3}{3!} + \ldots$$
$$R(\omega,\theta) = e^{\hat{\omega}t}$$

where $\hat{\omega}$ is a skew-symmetric matrix representing $\omega \times$

Rodrigue's Formula

For a skew - symmetric $\hat{a}$, the following hold :

$\hat{a}^2 = aa^T - |a|I$

$\hat{a}^3 = -|a|^2 \hat{a}$

With this relation, we can simplify

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\theta + \frac{(\hat{\omega}\theta)^2}{2!} + \frac{(\hat{\omega}\theta)^3}{3!} + \ldots$$

into

$$e^{\hat{\omega}\theta} = I + (\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots)\hat{\omega} + (\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \ldots)\hat{\omega}^2$$
$$= I + \hat{\omega}\sin\theta + \hat{\omega}^2(1 - \cos\theta)$$

Spatial Displacements

$\mathbf{X} = [A]\mathbf{x} + \mathbf{d}$

If **c** was a fixed point $\mathbf{c} = [A]\mathbf{c} + \mathbf{d}$

or $[I - A]\mathbf{c} = \mathbf{d}$.

But [*I*-*A*] is singular because a rotation matrix [*A*] has 1 as an eigenvalue

No such real **c**

Spatial Displacements

Though a spatial displacement has no fixed points, there is a fixed line, called the *screw axis*, that has the same position in space before and after the displacement. Any point on the fixed line is constrained to move along the line. The direction of this line is the axis of rotation of $[A]$ given by Rodrigues' vector $\mathbf{b}$. To determine the position of this line, let $\mathbf{d}^*$ be the projection of the translation vector $\mathbf{d}$ onto a plane perpendicular to $\mathbf{b}$, and we seek the solution to the equation

$$[A]\mathbf{c} + \mathbf{d}^* = \mathbf{c}$$

This defines the pole **c** of the planar displacement which rotates around **b** and translate in the plane perpendicular to **b**

The line is $\mathbf{L} = \mathbf{c} + t\,\mathbf{b}$

Spatial Displacements

The displacement is reduced to a pure rotation around the line $\mathbf{L} = \mathbf{c} + t\,\mathbf{b}$ and translation along it with the amount $d\mathbf{s} = d - d^*$ where $\mathbf{s} = \mathbf{b}/|\mathbf{b}|$

To solve for **c**, replace [*A*] by Caylay's formula, and multiply by [*I-B*], then simplify:

$$[I-A]\mathbf{c} = \mathbf{d}^*$$

$$[B]\mathbf{c} = -(1/2)[I-B]\mathbf{d}^*$$

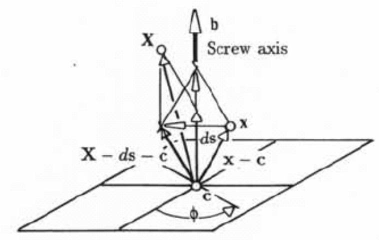
or

$$\mathbf{b} \times \mathbf{c} = -(1/2)(\mathbf{d}^* - \mathbf{b} \times \mathbf{d}^*)$$

This equation is simplified by operating on both sides with $\mathbf{b}\times$ and requiring that $\mathbf{b} \cdot \mathbf{c} = 0$, the result is

$$\mathbf{c} = (1/2)\left(\frac{\mathbf{b} \times \mathbf{d}^*}{\mathbf{b} \cdot \mathbf{b}} + \mathbf{d}^*\right)$$

Rodrigue’s Equation



For spatial displacement, we use **x-c** and **X-*d*s-c** instead of **x** and **X** resulting in

$$\mathbf{X} - (d\mathbf{s} + \mathbf{c}) - (\mathbf{x} - \mathbf{c}) = \mathbf{b} \times (\mathbf{X} - (d\mathbf{s} + \mathbf{c}) + \mathbf{x} - \mathbf{c}),$$

which simplifies to

$$\mathbf{X} - \mathbf{x} = \mathbf{b} \times (\mathbf{X} + \mathbf{x} - 2\mathbf{c}) + d\mathbf{s}.$$

Eigenvectors of 4x4 transforms

$$\begin{bmatrix} A & \mathbf{d} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \mathbf{p} \\ 1 \end{Bmatrix} = \lambda \begin{Bmatrix} \mathbf{p} \\ 1 \end{Bmatrix}$$

$$[T]\mathbf{p} = \lambda \mathbf{p}.$$

$$(1-\lambda)^2(\lambda^2 - 2\lambda\cos\phi + 1) = 0.$$