Algorithms for Constructing Immobilizing Fixtures and Grasps of Three-Dimensional Objects

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Abstract

We address the problem of computing immobilizing fixtures and grasps of three-dimensional objects, using simple fixturing devices and grippers with both discrete and continuous degrees of freedom. The proposed approach is based on the notion of second-order immobility introduced by Rimon and Burdick [48, 49, 50], which is used here to derive simple sufficient conditions for immobility and stability in the case of contacts between spherical locators and polyhedral objects. In turn, these conditions are the basis for efficient geometric algorithms that enumerate all of the stable immobilizing fixtures and grasps of polyhedra that can be achieved by various types of fixturing devices and grippers. Preliminary implementations of the proposed algorithms have been constructed, and examples are presented.

1 Introduction

We address the problem of immobilizing an object through a few contacts with simple modular fixturing elements, with applications in manufacturing (fixturing) and robotics (grasping). Our approach is based on the notion of second-order immobility introduced by Rimon and Burdick [48, 49, 50], and it is related to recent work in fixture planning by Wallack and Canny [57, 58], Brost and Goldberg [5], and Wagner, Zhuang, and Goldberg [56].

For concreteness, let us consider the fixturing device shown in Figure 1: it consists of two parallel plates with locator holes drilled along a rectangular grid, and of a set of spherical locators with integer height and radius. Four of these locators can be selected to form a fixture; either two of them are mounted on each plate (type I configuration, Figure 1(a)), or three locators are mounted on the first plate, the last one being mounted on the second plate (type II configuration, Figure 1(b)). The distance between the plates is a continuously adjustable degree of freedom of the device. (This device is a generalization of the two-dimensional fixturing vise proposed by Wallack and Canny [57, 58].)

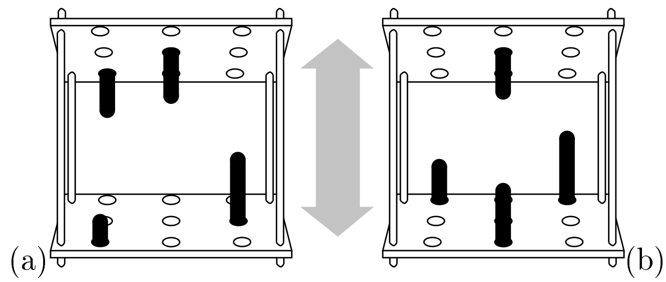

Figure 1: The proposed fixturing device: (a) type I configuration, (b) type II configuration.

Our goal is to compute the locator configurations (i.e., placements and heights) as well as the plate separation that will guarantee that a polyhedral part in *frictionless* contact with the locators is immobilized. To solve this problem, we must (1) formulate operational conditions for immobility, (2) enumerate all of the locator configurations that may achieve immobility, and (3) for each of these configurations, decide whether there exists a pose of the fixtured object that simultaneously achieves contact with the four locators and guarantees immobility. Our approach to step (1) is to specialize the conditions formulated by Rimon and Burdick to the class of fixturing elements and objects of interest. Step (2) can then be reduced to solving a combinatorial problem, so we can attack step (3) using numerical algebraic methods [28, 29, 37].

It should be noted that the conceptual design shown in Figure 1 can be implemented using standard modular fixturing elements such as the ones available in the QU CO kit: for example, a type I configuration can be constructed using two spherical locators mounted on a plate and two additional locators mounted on a beam clamp (Figure 2). A similar assembly with three locators and an adjustable vertical clamp can be used for type II configurations.

Although we will emphasize the fixturing problem in most of the paper, our approach applies to robotic grasp planning and other related problems (see Section 4). In particular, we are also developing an automatically reconfigurable gripper (Figure 3) which consists of two parallel plates whose distance can be adjusted by a computer-controlled actuator. The top plate carries a rectangular grid of individually-actuated locators, which can translate vertically under computer control. The bottom plate is a bare plane with three passive planar degrees of freedom, which are used to simulate the absence of friction and ensure grasp stability. This design is reminiscent of Goldberg's sliding-jaw gripper [17].

The rest of this paper is organized as follows. Previous work in fixture and grasp planning

Figure 2: Implementing a type I configuration of the proposed fixturing device using standard modules from the QU CO kit.

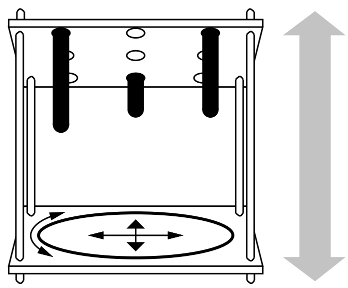

Figure 3: A reconfigurable gripper. An actuated locator is associated with each grid point. To avoid friction effects, the bottom plate should have three passive planar degrees of freedom.

is reviewed in Section 2. Our approach is described in Section 3: sufficient conditions for immobility and stability are derived in Section 3.1; they are used in Section 3.2 to design an efficient algorithm for enumerating all immobilizing stable fixtures of a given polyhedral object using the device shown in Figure 1. This algorithm is adapted to the problem of planning immobilizing grasps for the reconfigurable gripper of Figure 3 in Section 4, where some other extensions are also described. Preliminary experiments are presented in Section 5. Section 6 concludes with a brief discussion of our approach and future research directions.

2 Related Work

Modular fixturing systems consist of a kit of modules that can be reconfigured to fixture different parts. They have the potential of avoiding the costs associated with the design of custom fixtures, but pose the problem of automatically planning the module configurations adequate for a given part geometry. Traditionally, fixture designers have relied on heuristics such as the *3:2:1 fixturing principle* [19, 53]: the object to be fixtured is first positioned relative to a plane (*primary datum*) defined by three contact points; it is then positioned relative to a line (*secondary datum*) defined by two additional contact points, and finally positioned relative to a last point contact (*tertiary datum*). When the six points have been chosen correctly, the position of the fixtured object is completely determined as long as the contacts are maintained (*deterministic positioning* [1]). The object is then clamped into place by one or several additional contacts (*total constraint* [1]). Positioning is typically achieved through contact with *passive* fixturing elements such as plates, vee blocks, and locators, while clamping is achieved through contact with *active* fixturing elements, such as vises, toe clamps, or chucks.

The theoretical justification for such an approach finds its roots in the dual role of fixtures: immobilizing a part and resisting the forces and torques involved in manufacturing tasks such as assembly or machining.[1] Since screw theory [2, 22, 41] can be used to represented both displacements (*twists*) and forces and moments (*wrenches*), it is an appropriate tool for analyzing and designing fixtures. Indeed, it is known that six independent contact wrenches are necessary to prevent any infinitesimal displacement which maintains contact, and that a seventh one is required to ensure that contact cannot be broken (these correspond to the positioning and clamping contacts introduced above) [24, 54]. Such a fixture prevents any infinitesimal motion of the object, and it is said to achieve *form closure* [41, 47, 52]. A system of wrenches is said to achieve *force closure* when it can balance any external force and torque. Like wrenches and infinitesimal twists [51], force and form closure are dual notions and, as noted in [36, 39] for example, force closure implies form closure and vice versa.[2] In particular, fixtures achieving form/force closure also fulfill their second role as devices capable of resisting external forces and torques.

Past approaches to fixture planning have been based on expert systems [14, 18, 32], kinematic analysis and screw theory [1, 3, 8, 34], or a combination of both [12, 16]: Markus *et al.* have used a rule-based system to interactively design fixtures for box-type parts and to

[1] There are of course other issues involved in fixturing, for example the analysis of part deformation under clamping, see [10, 20] for approaches using finite-element methods.

[2] Note that there is unfortunately no general agreement on terminology in the grasping literature (see [55, 33] for discussions of this problem): for example, Reulaux [47], Salisbury [52], Ji [23], Markenscoff et al. [30] and Trinkle [55] use the expression form closure for what we call force closure, and reserve the expression force closure for grasps that can only balance certain external loads. Our definitions match the ones used by Mishra et al. [35], Nguyen [39], and Murray et al. [38].

select appropriate fixture modules [32]. Ferreira and Liu have used a generate-and-evaluate approach to determine the orientation of workpieces for machining operations [14]. Hayes and Wright have proposed *Machinist*, an expert-system-based process planner that incorporates fixturing information in the construction of a machining plan [18].

While expert systems are limited in their ability to generate fixture configurations based on analytical considerations, approaches based on screw theory can accurately predict the performance of fixture designs: Asada and By have proposed the *Automatically Reconfigured Fixturing (ARF)* system, which uses a detailed kinematic analysis to derive conditions for deterministic positioning, part accessibility and detachability, and total constraint [1]. Their approach has been integrated into a robotic assembly cell. Chou, Chandru, and Barash have developed a mathematical method based on screw theory for analyzing and synthesizing fixtures, and used linear programming to generate optimal clamp positions constrained to lie within convex contact polygons [8]. Bausch and Youcef-Toumi have introduced the notion of *motion stop* which represents the geometric resistance of a contact point to a given screw motion, and they have used it to compare fixture configurations. Their approach is integrated with the CATIA CAD system and is capable of synthesizing optimal fixture configurations from a discrete set of candidates.

Finally, it should be noted that some systems bridge the gap between expert systems and kinematic and force analysis: Gandhi and Thompson have proposed a methodology that relies on expert knowledge, force analysis, and geometric reasoning to synthesize and analyze modular fixture configurations [16]. Englert has also combined analytical considerations and knowledge-based methods to identify tradeoff relations between part production attributes and propose a control structure for planning part setup and clamping [12].

In the robotics community, the notions of form and force closure have been the basis of several approaches to grasp planning. Mishra, Schwartz, and Sharir [35] have proposed linear-time algorithms for computing a finger configuration achieving force closure for frictionless polyhedral objects. Markenscoff and Papadimitriou [31] and Mirtich and Canny [33] have proposed algorithms for planning grasps which are optimal according to various criteria [13]. In each of these works, the grasp-planning algorithm outputs a single grasp for a given set of contact faces. Assuming Coulomb friction [39], Nguyen has proposed instead a geometric method for computing *maximal independent* two-finger grasps of polygons, i.e., segments of the polygonal boundary where the two fingers can be positioned independently while maintaining force closure, requiring as little positional accuracy from the robot as possible. This approach has been generalized to handle various numbers of fingers and different object geometries in [4, 7, 42, 44, 45, 46]

Although robotic grasping and fixture planning are related (in both cases, the object grasped of fixtured must, after all, be held securely), their functional requirements are not the same: as remarked by Chou, Chandru, and Barash [8], machining a part requires much better positional accuracy than simply picking it up, and the range of forces exerted on the parts are very different. The role of friction forces is also different: in the grasping context, where fingers are often covered with rubber or other soft materials, friction effects can be used to lower the number of fingers required to achieve form closure from seven to four; in the fixturing context, on the other hand, it is customary to assume frictionless contact, partly due to the large magnitude and inherent dynamic nature of the forces involved [8] (see, however [27] for an approach to fixture planning with friction). Finally, the kinematic constraints on the positions of the contacts are also quite different.

Nonetheless, as noted by Wallack [58], there has recently been a renewed interest in the academic robotics community for manufacturing problems in general and fixturing in particular. Mishra has studied the problem of designing fixtures for rectilinear parts using toe clamps attached to a regular grid, and proven the existence of fixtures using six clamps [34] (this result has since then been tightened to four clamps by Zhuang, Goldberg, and Wong [59]). In keeping with the idea of *Reduced Intricacy Sensing and Control (RISC)* robotics of Canny and Goldberg [6], Wallack and Canny [57, 58] and Brost and Goldberg [5] have recently proposed very simple modular fixturing devices and efficient algorithms for constructing form-closure fixtures of two-dimensional polygonal and curved objects. Wagner, Zhuang, and Goldberg [56] have also proposed a three-dimensional seven-contact fixturing device and an algorithm for planning form-closure fixtures of a polyhedron with pre-specified pose.

All of the approaches discussed so far are based on the concepts of form and force closure. A different notion is *stability*: a part is said to be in stable equilibrium if it returns to its equilibrium position after having been subjected to a small displacement. Stability is very important in real mechanical systems which cannot be expected to have perfect accuracy. Nguyen has shown that force (or form) closure implies stability [40], but Donoghue, Howard and Kumar have shown that there exist stable grasps or fixtures which do not achieve form closure [11, 21]. Recently, Rimon and Burdick have introduced the notion of *second-order immobility* [48, 49, 50] and shown that certain equilibrium grasps (or fixtures) of a part which do not achieve form closure effectively prevent any *finite* motion of this part through curvature effects in configuration space. They have given operational conditions for immobilization and proven the dynamic stability of immobilizing grasps under various deformation models [50]. An additional advantage of this theory is that second-order immobilization can be achieved with fewer fingers (four contacts for convex fingers) than form closure (seven contacts [24, 54]). As detailed in the next section, we propose to exploit second-order immobility in fixture planning for three-dimensional polyhedral objects.

3 Proposed Approach

We first derive simple sufficient conditions for immobility and stability in the case of contacts between spherical locators and polyhedral objects (Section 3.1). We then use these conditions in Section 3.2 to design an efficient algorithm for planning stable immobilizing fixtures of a polyhedral object with the device shown in Figure 1.

3.1 Sufficient Conditions for Immobility and Stability

A sufficient Condition for Immobility. Let us consider a rigid object and the contacts between d locators and this object. Let us also denote by $\boldsymbol{p}_i$ $(i = 1, .., d)$ the positions of the contacts in a coordinate frame attached to the object, and by $\boldsymbol{n}_i$ $(i = 1, .., d)$ the unit inward normals to the corresponding faces.

We say that equilibrium is achieved when the contact wrenches balance each other, i.e.,

$$\sum_{i=1}^{d} \lambda_i \begin{pmatrix} \boldsymbol{n}_i \\ \boldsymbol{p}_i \times \boldsymbol{n}_i \end{pmatrix} = 0, \tag{1}$$

for some $\lambda_i \geq 0$ $(i = 1, .., d)$ with $\sum_{i=1}^{d} \lambda_i = 1$. Equilibrium is a necessary, but not sufficient, condition for force and form closure.

Czyzowicz, Stojmenovic and Urrutia have recently shown that three contacts in the plane and four contacts in the three-dimensional case are sufficient to immobilize (i.e., prevent any *finite* motion of) a polyhedron [9]. Rimon and Burdick have formalized the notion of immobilizing grasps and fixtures in terms of isolated points of the free configuration space [48, 49, 50]. They have shown that equilibrium fixtures that do not achieve form closure may still immobilize an object through second-order (curvature) effects in configuration space: a sufficient condition for immobility is that the relative curvature form associated with an essential equilibrium [3] grasp or fixture and defined by

$$\kappa_{\rm rel} = \sum_{i=1}^{d} \lambda_i |\boldsymbol{w}_i| \kappa_i$$

be negative definite. Here the weights λ_i are the equilibrium weights of (1), $|\boldsymbol{w}_i|$ is the magnitude of the wrench exerted by locator number i, and κ_i is the curvature form associated with the corresponding contact; this quadratic form is defined by:

$$\kappa_i = \frac{1}{|\boldsymbol{w}_i|}(\boldsymbol{v}^T, \boldsymbol{\omega}^T)(\mathcal{C}_i^T \mathcal{L}_i \mathcal{C}_i + \mathcal{D}_i)\begin{pmatrix}\boldsymbol{v}\\ \boldsymbol{\omega}\end{pmatrix},$$

where $\boldsymbol{v}$ and $\boldsymbol{\omega}$ denote the translational and rotational parts of an infinitesimal twist, $\mathcal{L}_i$ is a matrix related to the surface curvatures of the body and locator at the contact points, and the matrices $\mathcal{C}_i$ and $\mathcal{D}_i$ depend only on the position $\boldsymbol{p}_i$ of the contact point and the normal to $\boldsymbol{n}_i$ to the body's surface in $\boldsymbol{p}_i$.

Specializing the above equations to equilibrium contacts between spherical locators and polyhedra yields:

$$\kappa_{\rm rel} = \sum_{i=1}^{d} \lambda_i \boldsymbol{\omega}^T \mathcal{K}_i \boldsymbol{\omega}, \quad \text{where} \tag{2}$$

$\mathcal{K}_i = ([\boldsymbol{n}_{i\times}]^T[\boldsymbol{p}_{i\times}])^S - r_i[\boldsymbol{n}_{i\times}]^T[\boldsymbol{n}_{i\times}]$,

where r_i denotes the locator's radius, and by definition, $\mathcal{A}^S = \frac{1}{2}(\mathcal{A} + \mathcal{A}^T)$

Note that there is no term involving the translation $\boldsymbol{v}$ in this case. It follows from (2) that a sufficient condition for immobility is that the 3×3 symmetric matrix

$$\mathcal{K} = \sum_{i=1}^{d} \lambda_i \mathcal{K}_i$$

is negative definite.

A Sufficient Condition for Stability. We prove a sufficient condition for the stability (in the sense of Nguyen [40], see also [11, 21]) of a fixture configuration, and show that it is equivalent to the immobility condition derived in the previous paragraph (see [50] for a more general statement of the dynamic stability of immobilizing grasps).

[3] Essential equilibrium is achieved when the coefficients λ_i in (1) are uniquely defined and strictly positive [49].

Each locator is modeled as a sphere of radius r_i attached to a linear spring whose axis is aligned with the inward normal to the corresponding contact face. As the solid moves, the sphere slides on the contact face and translates along the corresponding spring (Figure 4).

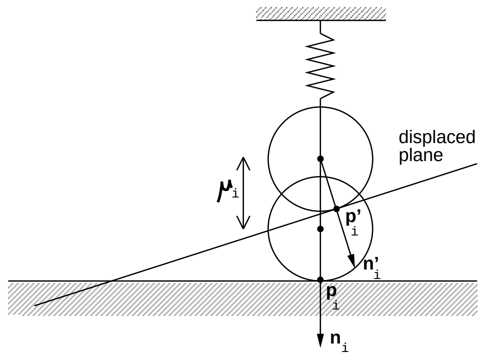

Figure 4: Small displacement of a plane in contact with a sphere mounted on a spring.

The potential energy of the fixture is the sum of the potential energies of the individual springs, i.e.,

$$U = \sum_{i=1}^{d} \frac{1}{2}\sigma_i^2,$$

where σ_i is the displacement of the spring associated with locator number i from its position at rest (we assume a unit spring constant for each locator). In general we can write $\sigma_i = -\lambda_i + \mu_i$, where λ_i is the compression at equilibrium (which is of course equal to the coefficient λ_i in (1)) and μ_i denotes the displacement of the sphere along $\boldsymbol{n}_i$ corresponding to a given displacement of the solid. An equilibrium fixture will be stable when it corresponds to a local minimum of the potential energy (as a function of small displacements of the object).

A rigid displacement is specified by a rotation $\mathcal{R}$ of axis $\boldsymbol{a}$ and angle θ, and a translation $\boldsymbol{v}$. Following Nguyen [40], we use a second-order Taylor expansion of the exponential definition of rotations, and parameterize μ_i by the twist $(\boldsymbol{v}, \boldsymbol{\omega})$, where $\boldsymbol{\omega} = \theta\boldsymbol{a}$.

The gradient and Hessian of the potential energy are respectively

$$\nabla U = \sum_{i=1}^{d} (-\lambda_i + \mu_i)\nabla\mu_i \quad \text{and} \tag{3}$$

$\nabla^2 U = \sum_{i=1}^{d} \nabla\mu_i \nabla\mu_i^T + (-\lambda_i + \mu_i)\nabla^2\mu_i$.

A simple calculation shows that the gradient of μ_i at the origin is the wrench $(\boldsymbol{n}_i^T, \boldsymbol{p}_i \times \boldsymbol{n}_i^T)^T$, and it follows as expected that the fixture is in equilibrium if and only if equation (1) is satisfied. To decide whether the equilibrium is stable, we must examine the Hessian of the potential function. Computing the Hessian of μ_i at the origin and substituting in (3) yields:

$$\nabla^2 U|_{0,0} = \mathcal{F} + \mathcal{S}, \quad \text{where}$$

$$\mathcal{F} = \sum_{i=1}^{d} \begin{pmatrix} \boldsymbol{n}_i \\ \boldsymbol{p}_i \times \boldsymbol{n}_i \end{pmatrix} \begin{pmatrix} \boldsymbol{n}_i \\ \boldsymbol{p}_i \times \boldsymbol{n}_i \end{pmatrix}^T, \quad \text{and}$$

$$\mathcal{S} = -\sum_{i=1}^{d} \lambda_i \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{K}_i \end{pmatrix}.$$

The equilibrium is stable when $\mathcal{F} + \mathcal{S}$, the Hessian of the potential function, or *stiffness matrix*, is positive definite. The matrix $\mathcal{F}$ is of course positive semi-definite, its zeros being the twists reciprocal to the wrenches exerted by the locators (which are guaranteed to exist for frictionless fixtures when $d \leq 6$). These twists satisfy the equations

$$\boldsymbol{v} \cdot \boldsymbol{n}_i + \boldsymbol{\omega} \cdot (\boldsymbol{p}_i \times \boldsymbol{n}_i) = 0 \quad \text{for} \quad i = 1, \ldots, d. \tag{4}$$

For equilibrium fixtures, only three of the above equations are independent, and for any choice of $\boldsymbol{\omega}$ there exists in general a vector $\boldsymbol{v}$ such that (4) is satisfied. Thus $\nabla^2 U$ is positive definite if and only if the matrix $\mathcal{K}$ is negative definite. This condition is the same as the sufficient condition for immobility derived earlier.

3.2 Planning Four-Locator Immobilizing Fixtures of Polyhedral Objects

In this section, we focus on the four-locator case and present an efficient algorithm for enumerating all immobilizing fixtures of a polyhedral object that can be achieved with the device of Figure 1. To simplify this planning process, we reduce the problem of achieving contact between a spherical locator and a plane to the problem of achieving point contact with a plane. This is done without loss of generality by growing the object to be fixtured by the locator radius and shrinking the spherical end of the locator into its center (see [5, 57, 58] for similar approaches in the two-dimensional case). For the sake of conciseness, we restrict our attention here to type II fixture configurations. Planning type I configurations involves analogous methods and has the same cost

The algorithm can be summarized as follows. For each quadruple of faces do:

1. Test whether they can be held in essential equilibrium.

2. Enumerate all locator configurations potentially achieving equilibrium through contacts with the selected faces.

3. For each such configuration, compute the pose of the object and test the immobilization condition.

Testing Essential Equilibrium For a polyhedral object, the normals $\boldsymbol{n}_i$ are fixed vectors. To ensure essential equilibrium, we restrict our attention to quadruples of faces such that no three of them have coplanar normals. This ensures that the coefficients λ_i in (1) are uniquely defined, and it allows us to compute them from the equation $\sum_{i=1}^{4} \lambda_i \boldsymbol{n}_i = 0$ and to test whether they all have the same sign. If they do not, the four candidate faces are rejected. If they do, we obtain three independent *linear* constraints on the positions of the locators on the faces:

$$\sum_{i=1}^{4} (\lambda_i \boldsymbol{n}_i) \times \boldsymbol{p}_i = 0. \tag{5}$$

(Note that the coefficients λ_i are now constants depending only on the choice of faces.)

We can parameterize each contact $\boldsymbol{p}_i$ by two variables u_i, v_i. Assuming convex faces, the fact that the contact points actually belong to the faces can be written as a set of linear inequalities on u_i, v_i:

$$f_{ij}(u_i, v_i) \leq 0, \quad j = 1, .., k_i, \tag{6}$$

where k_i is the number of edges that bound face number i.

Given a choice of four faces, a necessary and sufficient condition for the existence of contact points within those faces which achieve equilibrium is that there exists a solution to (5) subject to the constraints (6). This can be tested using linear programming. If the test is negative, the quadruple of faces is rejected.

For quadruples passing this second test, there is only (in general) a subset of each face that can participate in an equilibrium configuration. The subset corresponding to face number i is determined by projecting the five-dimensional polytope defined by (5) and (6) onto the plane (u_i, v_i). Several algorithms can be used to perform this projection, including Fourier's method [15], the convex hull and extreme point approaches of Lassez and Lassez [26, 25], and the Gaussian elimination and contour tracking techniques of Ponce *et al.* [46].

For faces with a bounded number of edges, all of these algorithms run in constant time, and they can be used to construct sub-faces that can be passed as input to the rest of the algorithm.

Enumerating Locator Configurations An exhaustive search of all possible grid coordinates would be extremely expensive: consider an object of diameter D (measured in units equal to the distance between successive plate holes); there are $O(D^8)$ type II possible configurations: one locator is at the origin with zero length, two locators have three integer coordinates, the last locator has only two. A similar line of reasoning also applies to type I configurations, and it yields the same order of complexity.

This has prompted us, like Wallack and Canny [57, 58] and Brost and Goldberg [5] in the two-dimensional case, to use bounds on the distance between two faces to restrict the set of grid coordinates under consideration. The minimum and maximum distances between pairs of points belonging to two given faces can be computed in constant time as follows: the maximum distance between two faces is always achieved for a pair of vertices. The minimum distance, on the other hand, may be achieved for any pair of face, edge, or vertex points (Figure 5). The first two cases shown in Figure 5 (face-face and edge-face pairs) only occur when two faces are parallel or when one edge is parallel to a face, and they reduce to computing the distance between a vertex and a face. Thus, there are only three non-trivial cases: the vertex-face, edge-edge, and vertex-edge pairs, and the corresponding distances are easily computed by constructing the unique straight line orthogonal to the pair of features of interest.

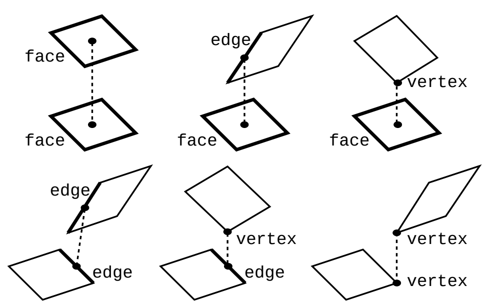

Figure 5: A list of the feature combinations yielding the minimum distance between two faces.

Let us position the first locator at the origin with zero length. The integer point corresponding to the second locator is then constrained to lie within the spherical shell centered at the origin with inner radius equal to the minimum distance between the two corresponding faces and outer radius equal to the maximum distance. Given the position of the second locator, the third locator is now constrained to lie within the region formed by the intersection of the two shells associated with the first and second locator. Finally, given the position of the third locator, the fourth locator is constrained to lie within a region formed by the intersection of three shells. The projection of this region onto the first plate yields the set of integer coordinates of the locator. Its last coordinate δ is determined at the next stage of the algorithm.

Computing the Pose Associated with a Given Locator Configuration. To avoid imposing a particular parameterization of the object's pose, we take advantage of the fact that a tetrahedron is completely determined by the lengths of its six edges.

We define the tetrahedron whose vertices are the four contacts by six quadratic equality constraints of the form

$$|\boldsymbol{p}_i - \boldsymbol{p}_j|^2 = l_{ij}^2, \text{ with } i = 1, 2, 3 \text{ and } i < j \leq 4, \tag{7}$$

which specify the lengths of the tetrahedron's edges.

At this point, the integer grid coordinates of the locators are fixed, and the coefficients l_{ij} are only functions of the variable δ. Thus the equalities (5) and (7) form a system of nine equations in the nine unknowns u_i, v_i $(i = 1, .., 4)$ and δ. Since three of these equations are linear, and the remaining six are quadratic, this system admits at most $2^6 = 64$ solutions which can easily be computed using homotopy continuation [37] or the toolkit for algebraic computation described in [28].

Once the solutions have been found, we can check whether they satisfy the linear inequalities (6) defining the contact faces, then check whether they achieve immobilization. Note that the object pose corresponding to a given locator configuration and plate distance is easily computed: since one of the locators is at the origin, we only need to compute the rotation mapping the fixturing device's coordinate frame onto the object's coordinate frame. Since we know the positions of the contacts in both coordinate systems, it is a simple matter to compute the corresponding rotation.

Algorithm Analysis. For each quadruple of faces, enumerating the locator configurations amounts to determining the integer positions falling in regions defined by the intersection of two, four, or six half-spaces bounded by spheres. A naive approach to that problem is to test every grid point against the constraints defining the regions of interest with cost $O(D^8)$, where D is as before the diameter of the object measured in units equal to the distance between two successive holes.

A better approach is to use a three-dimensional scan-line conversion algorithm to determine the integer points within a region in (optimal) time proportional to the number V of these points: scan-line conversion algorithms are used in computer graphics to render polygonal and curved shapes by enumerating pixels within these shapes one row at a time; they only require the ability to trace the shape boundaries and find their extrema in the horizontal and vertical dimension, and they have a time complexity linear in the size of their output. It is relatively straightforward to generalize these algorithms to the three-dimensional case: we can construct an explicit representation of the region boundaries by a procedure akin to boundary evaluation in constructive solid geometry. This process is simplified by the fact that in our case the boundary elements are sphere patches, circular arcs (intersections of two spheres), and vertices (intersections of three spheres). Constructing the boundary representation and its extrema in any direction can be done in constant time (given our bounded number of half-spaces), and scan-line conversion can then proceed, one plane at a time, in time proportional to V (Figure 6). Thus, the time complexity of our overall algorithm is $O(N^4V)$ where N is the number of faces of the polyhedron. Of course, V is still, in the worst case, $O(D^8)$.

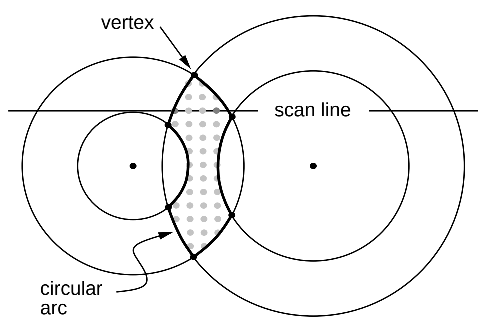

Figure 6: Illustration of scan-line conversion in the 2D case: spans between consecutive boundary elements are filled one scan-line at a time.

To obtain a more realistic estimate of the algorithm behavior, we can parameterize V by the diameter D of the object and the maximum range between the minimum and maximum distance between two faces, say r. If we assume that $r \ll D$ and that the distance between two faces is bounded below by some strictly positive number,[4] then the regions corresponding to the possible positions of the second, third, and fourth locator have respectively the "thick" spherical, circular, and punctual shapes shown in Figure 7. The corresponding volumes are respectively $O(D^2 r)$, $O(Dr^2)$, and $O(r^3)$. The projection of the latter volume on the plate has an area of $O(r^2)$, and it follows that $V = O(D^3 r^5) \ll O(D^8)$. Experiments will allow us to conduct an empirical evaluation of this model.

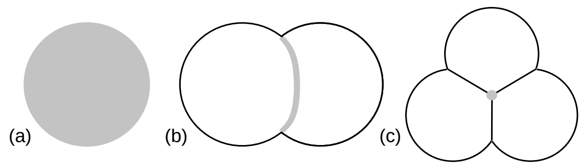

Figure 7: The regions shown in grey correspond to the position of: (a) the second locator, (b) the third locator, (c) the fourth locator.

4 Extensions

In this section, we show that our general approach to fixturing applies to a variety of related problems. For the sake of conciseness, we do not go into as much detail as in the previous section.

4.1 Planning Immobilizing Grasps for a Reconfigurable Gripper

Consider the reconfigurable gripper shown in Figure 3: it is similar to the device of Figure 1, except that the bottom plate is just a bare plane, and that the top plate carries a rectangular array of individually-actuated locators. This gripper can be used to immobilize a polyhedral object through contacts with three of the top plate locators, and either a face, an edge-and-vertex, or a three-vertex contact with the bottom plate. Let us assume for the sake of simplicity that the faces of the polyhedron are triangular (convex faces can be handled in similar ways, see [8] for a related approach). Any wrench exerted at a contact point between a face and the bottom plate can be written as a positive combination of wrenches at the vertices. Likewise, the wrenches corresponding to an edge-and-vertex contact are positive combinations of wrenches exerted at the end-points of the line segment and at the

[4]This is of course not true in general for an arbitrary polyhedron; however in practice two contact points are separated by at least one unit of distance.

vertex. Thus equilibrium configurations can be found, in general, by writing the equilibrium equation (1) for six elementary wrenches.

We detail the case of a contact between the bottom plate and a triangular face with unit normal $\boldsymbol{n}$ and vertices $\boldsymbol{v}_i$ $(i = 1, 2, 3)$. Let $\boldsymbol{p}_i$ and $\boldsymbol{n}_i$ $(i = 1, 2, 3)$ denote the remaining contact points and surface normals; we take advantage of the fact that the overall scale of the wrenches is irrelevant to rewrite (1) as

$$\sum_{i=1}^{3} \lambda_i \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{v}_i \times \boldsymbol{n} \end{pmatrix} + \sum_{i=1}^{3} \mu_i \begin{pmatrix} \boldsymbol{n}_i \\ \boldsymbol{p}_i \times \boldsymbol{n}_i \end{pmatrix} = 0, \quad \text{with} \tag{8}$$

$\lambda_1 + \lambda_2 + \lambda_3 = 1$, and $\lambda_i, \mu_i \geq 0$ $(i = 1, 2, 3)$.

When the four surface normals are linearly independent, the equation $\boldsymbol{n} = -\sum_{i=1}^{3} \mu_i \boldsymbol{n}_i$ allows us to compute the coefficients μ_i and check whether they have the same sign. If they do not, the quadruple of faces under consideration is rejected. If they do, (8) provides four linear equations in the the nine unknowns λ_i, u_i, v_i $(i = 1, 2, 3)$, where u_i, v_i parameterize as before the position of the contact point $\boldsymbol{p}_i$ within the corresponding face. We test the existence of equilibrium configurations by using linear programming to determine whether the polytope defined by the constraints (6), (8), and $\lambda_i \geq 0$ $(i = 1, 2, 3)$ is empty. When this polytope is not empty, we determine as before the subset of each face that may participate in an equilibrium configuration by projecting it onto the plane u_i, v_i $(i = 1, 2, 3)$. Ideas similar to the ones used in the four-locator case can be used to reduce the subset of integer locator positions that needs to be considered.

We now show how to compute the pose of the object for a given configuration of the locators. We write that the difference in height between the contact points $\boldsymbol{p}_j$ and $\boldsymbol{p}_1$ $(j = 1, 2)$ is equal to the difference in height δ_j between the corresponding locators. This yields two linear equations

$$(\boldsymbol{p}_j - \boldsymbol{p}_1) \cdot \boldsymbol{n} = \delta_j, \quad j = 1, 2, \tag{9}$$

in the unknowns u_i, v_i $(i = 1, 2, 3)$.

We still need three additional equations to compute the pose of the object. Let us denote by $\boldsymbol{q}_i$ $(i = 1, 2, 3)$ the projections of the three locators onto the plane of the bottom plate. The planar transformation mapping the vector $\boldsymbol{q}_2 - \boldsymbol{q}_1$ onto the vector $\boldsymbol{q}_3 - \boldsymbol{q}_1$ is the linear map $\mathcal{T}$ obtained by composing the rotation that aligns the two vectors with the scaling that gives them the same length. The transformation $\mathcal{T}$ is trivially determined from the points $\boldsymbol{q}_i$.

Let us denote by $\boldsymbol{p}_i' = \boldsymbol{p}_i - (\boldsymbol{p}_i \cdot \boldsymbol{n})\boldsymbol{n}$ the projection of the point $\boldsymbol{p}_i$ $(i = 1, 2, 3)$ onto the plane of the bottom plate. We now write that the two triangles formed respectively by the points $\boldsymbol{p}_i'$ and $\boldsymbol{q}_i$ are similar. This can be expressed by the linear vector equation

$$(\boldsymbol{p}_3' - \boldsymbol{p}_1') = \mathcal{T}(\boldsymbol{p}_2' - \boldsymbol{p}_1'), \tag{10}$$

which expresses the fact that the two triangles are homotetic, and by the quadratic equation

$$|\boldsymbol{p}_2' - \boldsymbol{p}_1'|^2 = |\boldsymbol{q}_2 - \boldsymbol{q}_1|^2, \tag{11}$$

which expresses the fact that the two triangles have the same size.

Together, (8), (9) and (10) form a system of eight linear equations that can be used to solve for eight of the unknowns as a function of the remaining one. Substituting in (10) finally yields a univariate quadratic equation which is readily solved.

It follows that we can enumerate as before the equilibrium configurations, then test them for immobility. For planar contacts, equilibrium will guarantee immobilization, while for edge-and-vertex contacts a conservative condition will be to use point contact ($r_i = 0$) in the expression of the relative curvature form. The complexity of the algorithm is $O(N^4 D^6)$. Interestingly, the solution of a six-contact fixturing problem has a lower complexity than the four-contact fixturing problem encountered before.

4.2 Planning Three-Locator Stable Equilibrium Fixtures

Consider the problem of computing "fixture" configurations in which a polyhedral object rests in a stable fashion on three locators with integer lengths attached to a single horizontal plate (Figure 8).[5] Here we take the effect of gravity into account, and assume that the vertical is aligned with one of the axes of the grid. The method is similar to the earlier one.

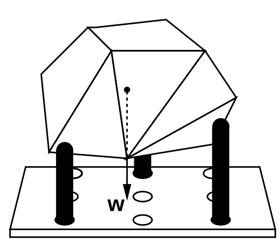

Figure 8: A three-locator "fixturing" device.

[5]This problem was suggested by Ken Goldberg.

We choose the center of mass of the object as the origin. Writing that the weight $\boldsymbol{w}$ and the reaction forces achieve equilibrium yields

$$\begin{pmatrix} \boldsymbol{w} \\ 0 \end{pmatrix} + \sum_{i=1}^{3} \lambda_i \begin{pmatrix} \boldsymbol{n}_i \\ \boldsymbol{p}_i \times \boldsymbol{n}_i \end{pmatrix} = 0, \tag{12}$$

with $\lambda_i \geq 0$ for for $i = 1, 2, 3$.

We can assume unit weight, and by choosing one of the locators as the origin of the grid, we obtain six more quadratic constraints:

$$|\boldsymbol{w}|^2 = 1, \quad \begin{cases} |\boldsymbol{p}_1 - \boldsymbol{p}_2|^2 = l_{12}^2 \\ |\boldsymbol{p}_2 - \boldsymbol{p}_3|^2 = l_{23}^2 \\ |\boldsymbol{p}_3 - \boldsymbol{p}_1|^2 = l_{31}^2 \end{cases}, \quad \text{and} \tag{13}$$

$$\begin{cases} (\boldsymbol{p}_2 - \boldsymbol{p}_1) \cdot \boldsymbol{w} = k_1 \\ (\boldsymbol{p}_3 - \boldsymbol{p}_1) \cdot \boldsymbol{w} = k_2 \end{cases}.$$

Note that the last two equations are necessary to ensure that the vertical is aligned with one of the grid axes. Together, equations (12) and (13) yield nine quadratic constraints in the nine variables λ_i, u_i, v_i $(i = 1, 2, 3)$ (it is not necessary to explicitly use $\boldsymbol{w}$ as an unknown, since the vector $-\sum_{i=1}^{3} \lambda_i \boldsymbol{n}_i$ can be used instead). The total degree of the system is $2^9 = 512$. Once the solutions of the equations have been found, they can be checked to see whether the corresponding contacts actually lie inside the faces and whether the equilibrium is stable (of course immobilization is not achieved anymore in this case). It should be noted that planning the position of three locators instead of four reduces the complexity of planning to $O(N^3 D^6)$. Another important point is that for the device of Figure 8 to work properly, it is very important that no friction occurs at the contacts. This can be achieved by mounting the locator caps on ball bearings, at the cost of some loss in positioning accuracy.

4.3 Four-Finger Immobilizing Grasp Planning

We now consider the problem of planning a four-finger immobilizing grasp of a polyhedral object. First, it should be obvious that when the positions of the contact points between fingers and object faces are unconstrained, equations (5) and (6) form a set of *linear* constraints defining the set of all equilibrium grasps. Thus this set can be completely characterized through linear programming. As shown earlier, an equilibrium grasp achieves immobilization when the matrix $\mathcal{K}$ is negative definite. For a given choice of faces, the only unknowns in the matrix $\mathcal{K}$ are the finger positions $\boldsymbol{p}_i$. However, testing for definite negativeness involves cubic constraints on the elements of $\mathcal{K}$.

As shown in [46], four-finger equilibrium grasps can be divided into three sub-classes: concurrent, two-pencil, and regulus grasps (Figure 9).[6] Thus we can examine each class separately: by specializing for each of them the expression of $\mathcal{K}_i$ given in (2) and, if appropriate, the twist reciprocity condition given in (4), we will simplify the form of the immobility condition with the goal of finding linear sufficient conditions for immobility.

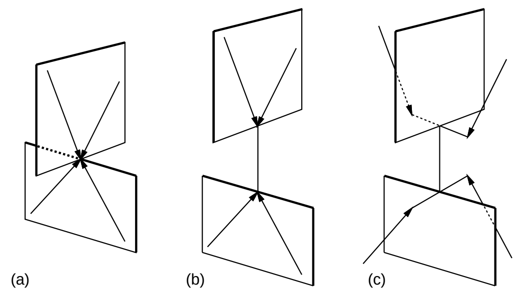

Figure 9: Classes of grasps achieving equilibrium: (a) the contact forces intersect in one point; (b) they form two non-coplanar pencils; (c) they form a regulus. The contact faces are not shown.

We illustrate this process with concurrent grasps, such that the lines of action of the four forces intersect in one point (Figure 9(a)). These grasps can be parameterized by

$$\boldsymbol{p}_i = \boldsymbol{p}_0 - \rho_i \boldsymbol{n}_i, \tag{14}$$

where $\boldsymbol{p}_0$ is the point where the lines of action of the forces intersect, and ρ_i is the signed distance between $\boldsymbol{p}_0$ and $\boldsymbol{p}_i$, such that $\rho_i > 0$ if $\boldsymbol{p}_0$ is on the interior side of face number i. Substituting in (2) and taking advantage once again of the fact that, at equilibrium, $\sum_{i=1}^4 \lambda_i \boldsymbol{n}_i = 0$, we obtain immediately

$$\mathcal{K} = -\sum_{i=1}^{4} \lambda_i(\rho_i + r_i)[\boldsymbol{n}_{i\times}]^T[\boldsymbol{n}_{i\times}].$$

A sufficient (but not necessary) condition for $\mathcal{K}$ to be definite negative is of course that

$$\rho_i + r_i > 0, \quad \text{for} \quad i = 1, \ldots, 4, \tag{15}$$

and it follows that the immobilizing grasps satisfying this condition can be completely characterized through linear programming under the constraints (5), (6), and (15). Since the

[6] Of course, four co-planar forces can also achieve equilibrium, but not essential equilibrium.

stiffness matrix of the grasp is determined by $\mathcal{K}_i$, we can also compute optimal grasps by maximizing $\min_i(\rho_i + r_i)$ under these constraints.

We plan to seek similar linear sufficient conditions for the other two types of equilibrium grasps.

5 Implementation and Results

We have fully implemented the four-locator fixturing algorithm and the algorithm for planning immobilizing grasps using the reconfigurable gripper proposed in Section 4.1. The fixtures have been physically implemented using QU CO fixturing elements as shown in Figure 2. Because we are still in the process of constructing the reconfigurable gripper, we can only present simulated grasping experiments. Both implementations have been written in C, and they include the two pruning stages proposed in Section 3: the subsets of the faces that may participate in an immobilizing grasp are first found by projecting the polytope defined by the equilibrium constraints (5) and (6) onto the parameter space of the faces. Candidate configurations that satisfy the distance constraints associated with these subsets of the faces are then enumerated by scan-converting the volumes bounded by the corresponding spherical shells.

5.1 Four-Locator Fixture Planning

As shown in Figure 2, we have constructed the proposed four-locator fixturing device using modular fixturing elements from the QU CO kit. We have used an aluminum base plate with an array of threaded holes, compatible threaded bolts and nuts, removable spherical locator tips, and a horizontal beam. The bolts and spherical tips are used as locators, and different locator heights are implemented by attaching different number of nuts to the bolts before screwing them through the threaded holes of the base plate. The horizontal beam is used as a support for the top locators.

Figure 10 shows some simulation results. The test object is a tetrahedron, and each result is shown from two different viewpoints. Figure 11 shows the fixturing device in the immobilizing configuration given in 10.a.

Table 1 shows some quantitative results for different grid resolutions. In our experiments we have used a $K \times K$ grid with various values of K, as well as locators whose height may

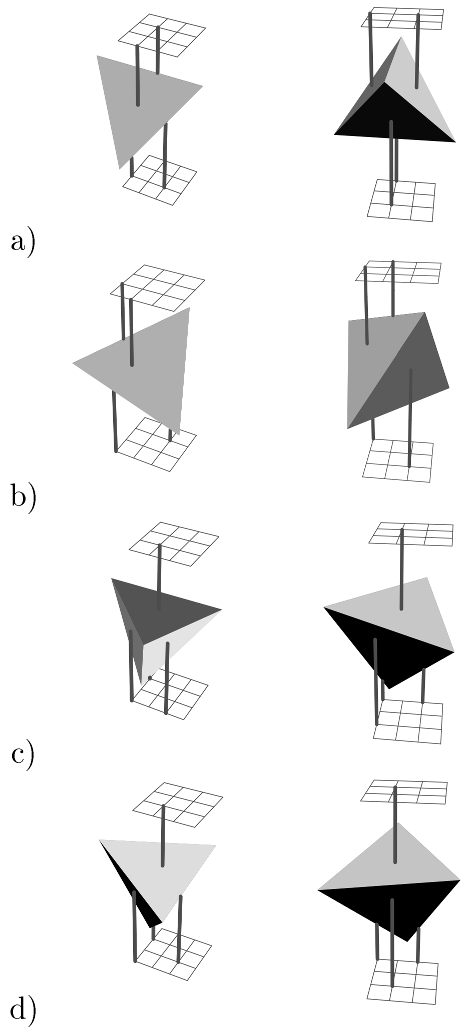

Figure 10: Some solutions for the four-locator fixturing device.

Figure 11: The fixturing device immobilizing a tetrahedron.

take ten discrete values. Table 1 shows the results obtained without any pruning (N), using spherical shell pruning only (S), and combining the projection- and shell-pruning stages (P+S).

K	# Candidates		
	N	S	P+S
3	6,377,292	267,868	223,224
4	63,700,992	4,429,772	3,601,440
5	379,687,500	20,720,018	17,709,408
6	1,632,586,752	297,104,432	237,683,544

Table 1: Quantitative results using a tetrahedron as a test object.

We have used homotopy continuation [37] to solve the polynomial system of degree 64 that determines the poses of an object compatible with a given locator configuration. Our distributed implementation of continuation takes roughly 2.5 seconds on two networked four-processor SUN SPARCstations 10 to solve this system. Thus we have not been able in our actual experiments with moderate grid resolutions to compute in a reasonable time all of the achievable fixtures. Instead, we have stopped the computation once a few immobilizing fixtures had been found. The statistics given in Table 1 have been obtained by running only the part of the algorithm that enumerates all possible locator configurations. We have recently found a new pose parameterization that only requires solving a system of degree 32 and are in the process of implementing it.

5.2 Planning Grasps for the Reconfigurable Gripper

In the case of the gripper, we have recently found that the orientation of an immobilized object is independent of the heights of the three locators, and that this orientation can be computed by solving a univariate quadratic equation. This has allowed us to construct an efficient implementation of our grasp planning algorithm. Figures 12 and 13 show some of the grasps found for a tetrahedron for two different grid resolutions, and Table 2 shows some quantitative results. In this case, we have been able to compute all of the immobilizing grasps; the run times reported in Table 2 were measured on a SUN SPARCstation 10.

Figures 14 amd 15 show some more simulation examples using a polyhedron with ten faces. As shown in Table 3, in this case pruning eliminates a much larger percentage of the

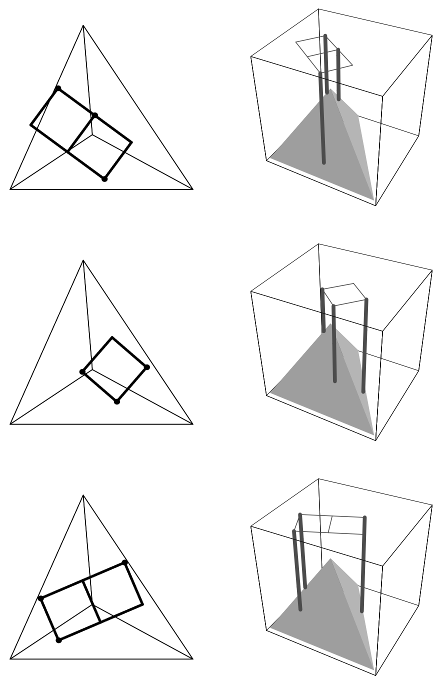

Figure 12: Grasping a tetrahedron: some solutions for a 5×5 grid.

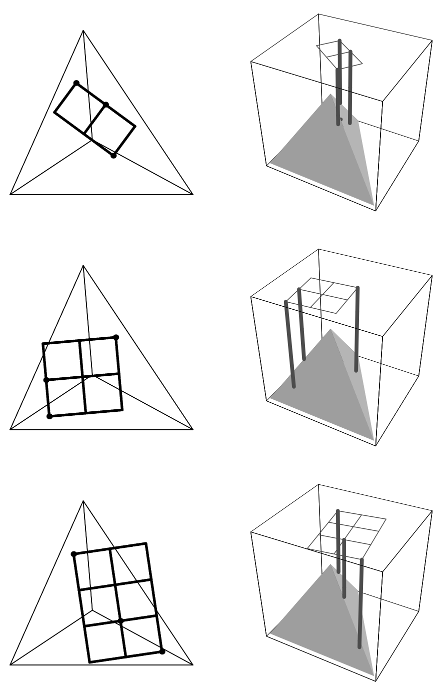

Figure 13: Some solutions for a 6×6 grid.

K	Number of Solutions	Run Time (s)			# Candidates		
		N	S	P+S	N	S	P+S
3	0	1	1	1	33	10	10
4	160	1	1	1	141	42	40
5	704	2	1	2	411	145	135
6	1,963	4	2	2	927	391	378
7	4,263	8	4	4	1,839	795	751

Table 2: Quantitative results using a tetrahedron as a test object.

possible configurations, corresponding to the fact that, for most choices of faces, the range between the minimum and maximum distances being smaller than in the previous case.

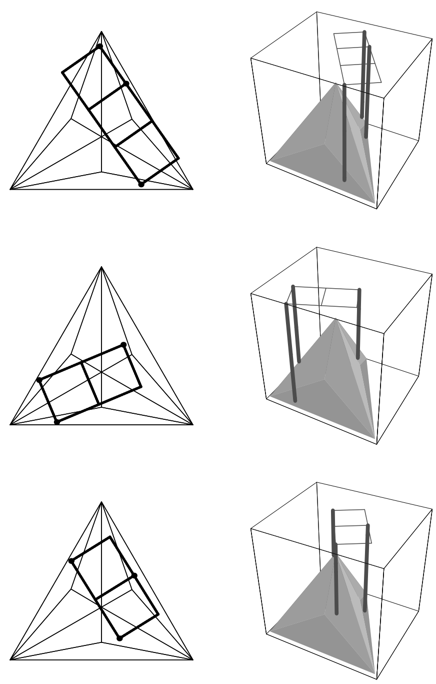

Figure 14: Grasping a 10-face polyhedron: some solutions for a 5×5 grid.

6 Discussion

We have proposed and implemented various algorithms for fixturing and grasping three-dimensional polyhedra. We are in the process of constructing a prototype of the automatically reconfigurable gripper whose conceptual design is shown in Figure 3.

There are obviously polyhedral objects which cannot be fixtured with our device (a trivial

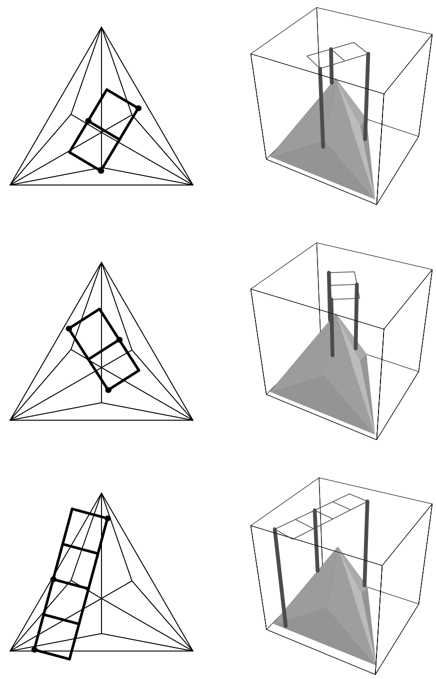

Figure 15: Some solutions for a 6×6 grid.

K	# Sol	Run Time (s)			# Candidates		
		N	S	P+S	N	S	P+S
3	0	20	1	2	2,772	750	712
4	189	47	3	4	11,844	2,213	2,102
5	794	72	9	9	34,524	3,819	3,537
6	2,326	142	20	20	77,868	7,811	7,125
7	5,046	341	43	41	154,476	16,259	14,951

Table 3: Quantitative results using a 10-face polyhedron as a test object.

example is an object whose diameter is smaller than the inter-locator distance). It would be interesting to characterize precisely the class of fixturable objects (see [59] for a discussion of the two-dimensional case).

Another interesting avenue of research would be to extend the proposed algorithm to parts bounded by algebraic patches (see [58, Chapter 6] for the two-dimensional case). Our overall approach extends to this case in a straightforward way, but working out the details of how to enumerate locator configurations and dealing with the very high degree of the equations involved should prove quite challenging.

Finally, we plan to investigate in-hand manipulation using our reconfigurable gripper.

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