**Def** Innerproduct

$$\equiv (\text{Insert } +)\circ(\text{ApplyToAll } \times)\circ\text{Transpose}$$

Or, in abbreviated form:

**Def** IP $\equiv (/+)\circ(\alpha\times)\circ$Trans.

| | |
|---|---|
| IP:<<1,2,3>, <6,5,4>> = | |
| Definition of IP | ⇒ (/+)∘(α×)∘Trans: <<1,2,3>, <6,5,4>> |
| Effect of composition, ∘ | ⇒ (/+):((α×):(Trans: <<1,2,3>, <6,5,4>>)) |

| | |
|---|---|
| Applying Transpose | ⇒ (/+):((α×): <<1,6>, <2,5>, <3,4>>) |
| Effect of ApplyToAll, α | ⇒ (/+): <×: <1,6>, ×: <2,5>, ×: <3,4>> |
| Applying × | ⇒ (/+): <6,10,12> |
| Effect of Insert, / | ⇒ +: <6, +: <10,12>> |
| Applying + | ⇒ +: <6,22> |
| Applying + again | ⇒ 28 |

### 11.2 Description

An FP system comprises the following:

1) a set O of *objects*;
2) a set F of *functions f* that map objects into objects;
3) an operation, *application*;
4) a set **F** of *functional forms*; these are used to combine existing functions, or objects, to form new functions in F;
5) a set D of *definitions* that define some functions in F and assign a name to each.

**Selector functions**

$1\!:\!x \equiv x{=}\langle x_1, \ldots, x_n\rangle \rightarrow x_1;\ \bot$

and for any positive integer s

$s\!:\!x \equiv x = \langle x_1, \ldots, x_n\rangle\ \&\ n \geq s \rightarrow x_s;\ \bot$

Thus, for example, $3\!:\!\langle A,B,C\rangle = C$ and $2\!:\!\langle A\rangle = \bot$. Note that the function symbols 1, 2, etc. are distinct from the atoms *1*, *2*, etc.

**Tail**

$\text{tl}\!:\!x \equiv x{=}\langle x_1\rangle \rightarrow \phi;$

$x{=}\langle x_1, \ldots, x_n\rangle\ \&\ n \geq 2 \rightarrow \langle x_2, \ldots, x_n\rangle;\ \bot$

**Identity**

$\text{id}\!:\!x \equiv x$

**Atom**

atom:$x \equiv x$ is an atom $\rightarrow T$; $x \neq \perp \rightarrow F$; $\perp$

**Equals**

eq:$x \equiv x = \langle y,z \rangle \;\&\; y = z \rightarrow T$; $x = \langle y,z \rangle \;\&\; y \neq z \rightarrow F$; $\perp$

**Null**

null:$x \equiv x = \phi \rightarrow T$; $x \neq \perp \rightarrow F$; $\perp$

**Reverse**

reverse:$x \equiv x = \phi \rightarrow \phi$;

$x = \langle x_1, \dots, x_n \rangle \rightarrow \langle x_n, \dots, x_1 \rangle$; $\perp$

**Distribute from left; distribute from right**

distl:$x \equiv x = \langle y,\phi \rangle \rightarrow \phi$;

$x = \langle y, \langle z_1, \dots, z_n \rangle \rangle \rightarrow \langle \langle y,z_1 \rangle, \dots, \langle y,z_n \rangle \rangle$; $\perp$

distr:$x \equiv x = \langle \phi,y \rangle \rightarrow \phi$;

$x = \langle \langle y_1, \dots, y_n \rangle, z \rangle \rightarrow \langle \langle y_1,z \rangle, \dots, \langle y_n,z \rangle \rangle$; $\perp$

**Length**

length:$x \equiv x = \langle x_1, \dots, x_n \rangle \rightarrow$ n; $x = \phi \rightarrow 0$; $\perp$

**Add, subtract, multiply, and divide**

$+$:$x \equiv x = \langle y,z \rangle \;\&\; y,z$ are numbers $\rightarrow y+z$; $\perp$

$-$:$x \equiv x = \langle y,z \rangle \;\&\; y,z$ are numbers $\rightarrow y-z$; $\perp$

$\times$:$x \equiv x = \langle y,z \rangle \;\&\; y,z$ are numbers $\rightarrow y \times z$; $\perp$

$\div$:$x \equiv x = \langle y,z \rangle \;\&\; y,z$ are numbers $\rightarrow y \div z$; $\perp$

(where $y \div 0 = \perp$)

**Transpose**

trans:$x \equiv x = \langle \phi, \dots, \phi \rangle \rightarrow \phi$;

$x = \langle x_1, \dots, x_n \rangle \rightarrow \langle y_1, \dots, y_m \rangle$; $\perp$

where

$x_i = \langle x_{i1}, \dots, x_{im} \rangle$ and

$y_j = \langle x_{1j}, \dots, x_{nj} \rangle$, $1 \leq i \leq n$, $1 \leq j \leq m$.

**And, or, not**

and:$x \equiv x = \langle T,T \rangle \rightarrow T$;

$x = \langle T,F \rangle \vee x = \langle F,T \rangle \vee x = \langle F,F \rangle \rightarrow F$; $\perp$

etc.

**Append left; append right**

apndl:$x \equiv x = \langle y,\phi \rangle \rightarrow \langle y \rangle$;

$x = \langle y, \langle z_1, \dots, z_n \rangle \rangle \rightarrow \langle y, z_1, \dots, z_n \rangle$; $\perp$

apndr:$x \equiv x = \langle \phi,z \rangle \rightarrow \langle z \rangle$;

$x = \langle \langle y_1, \dots, y_n \rangle, z \rangle \rightarrow \langle y_1, \dots, y_n, z \rangle$; $\perp$

**Right selectors; Right tail**

1r:$x \equiv x = \langle x_1, \dots, x_n \rangle \rightarrow x_n$; $\perp$

2r:$x \equiv x = \langle x_1, \dots, x_n \rangle \;\&\; n \geq 2 \rightarrow x_{n-1}$; $\perp$

etc.

tlr:$x \equiv x = \langle x_1 \rangle \rightarrow \phi$;

$x = \langle x_1, \dots, x_n \rangle \;\&\; n \geq 2 \rightarrow \langle x_1, \dots, x_{n-1} \rangle$; $\perp$

**Rotate left; rotate right**

rotl:$x \equiv x = \phi \rightarrow \phi$; $x = \langle x_1 \rangle \rightarrow \langle x_1 \rangle$;

$x = \langle x_1, \dots, x_n \rangle \;\&\; n \geq 2 \rightarrow \langle x_2, \dots, x_n, x_1 \rangle$; $\perp$

etc.

Functional Form

**Composition**
$(f \circ g):x \equiv f:(g:x)$

**Construction**
$[f_1, \ldots, f_n]:x \equiv <f_1:x, \ldots, f_n:x>$ (Recall that since $<\ldots, \bot, \ldots> = \bot$ and all functions are $\bot$-preserving, so is $[f_1, \ldots, f_n]$.)

**Condition**
$(p \rightarrow f; g):x \equiv (p:x)=T \rightarrow f:x; \quad (p:x)=F \rightarrow g:x; \bot$

**Constant** (Here $x$ is an object parameter.)
$\bar{x}:y \equiv y=\bot \rightarrow \bot; x$

**Insert**
$/f:x \equiv x=<x_1> \rightarrow x_1; x=<x_1, \ldots, x_n> \;\&\; n \geq 2$
$\rightarrow f:<x_1, /f:<x_2, \ldots, x_n>>; \bot$

If $f$ has a unique right unit $u_f \neq \bot$, where $f:<x,u_f> \in \{x, \bot\}$ for all objects $x$, then the above definition is extended: $/f:\phi = u_f$. Thus

$/+:<4,5,6> = +:<4, +:<5, /+:<6>>>$
$= +:<4, +:<5,6>> = 15$
$/+:\phi=0$

**Apply to all**
$\alpha f:x \equiv x=\phi \rightarrow \phi;$
$x=<x_1, \ldots, x_n> \rightarrow <f:x_1, \ldots, f:x_n>; \bot$

**Binary to unary** ($x$ is an object parameter)
$(\text{bu}\ f\ x):y \equiv f:<x,y>$

Thus

$(\text{bu} + 1):x = 1+x$

**While**
$(\text{while}\ p\ f):x \equiv p:x=T \rightarrow (\text{while}\ p\ f):(f:x);$
$p:x=F \rightarrow x; \bot$

Definition

**Def** $\text{last} \equiv \text{null} \circ \text{tl} \rightarrow 1; \text{last} \circ \text{tl}$

$\text{last}:\langle 1,2\rangle$:

| | |
|---|---|
| $\text{last}:\langle 1,2\rangle =$ | |
| definition of last | $\Rightarrow (\text{null}\circ\text{tl} \rightarrow 1;\ \text{last}\circ\text{tl}):\langle 1,2\rangle$ |
| action of the form $(p\rightarrow f;\ g)$ | $\Rightarrow \text{last}\circ\text{tl}:\langle 1,2\rangle$ |
| | since $\text{null}\circ\text{tl}:\langle 1,2\rangle = \text{null}:\langle 2\rangle = F$ |
| action of the form $f\circ g$ | $\Rightarrow \text{last}:(\text{tl}:\langle 1,2\rangle)$ |
| definition of primitive tail | $\Rightarrow \text{last}:\langle 2\rangle$ |
| definition of last | $\Rightarrow (\text{null}\circ\text{tl} \rightarrow 1;\ \text{last}\circ\text{tl}):\langle 2\rangle$ |
| action of the form $(p\rightarrow f;\ g)$ | $\Rightarrow 1:\langle 2\rangle$ |
| | since $\text{null}\circ\text{tl}:\langle 2\rangle = \text{null}:\phi = T$ |
| definition of selector 1 | $\Rightarrow 2$ |

**Def** $! \equiv \text{eq0} \rightarrow \bar{1};\ \times\circ[\text{id},\ !\circ\text{sub1}]$

where

**Def** $\text{eq0} \equiv \text{eq}\circ[\text{id},\ \bar{0}]$
**Def** $\text{sub1} \equiv -\circ[\text{id},\ \bar{1}]$

Here are some of the intermediate expressions an FP system would obtain in evaluating $!:2$:

$$
\begin{aligned}
!:2 &\Rightarrow (\text{eq0} \rightarrow \bar{1};\ \times\circ[\text{id},\ !\circ\text{sub1}]):2 \\
&\Rightarrow \times\circ[\text{id},\ !\circ\text{sub1}]:2 \\
&\Rightarrow \times:\langle \text{id}:2,\ !\circ\text{sub1}:2\rangle \Rightarrow \times:\langle 2,\ !:1\rangle \\
&\Rightarrow \times:\langle 2,\ \times:\langle 1,\ !:0\rangle\rangle \\
&\Rightarrow \times:\langle 2,\ \times:\langle 1,\bar{1}:0\rangle\rangle \Rightarrow \times:\langle 2,\ \times:\langle 1,1\rangle\rangle \\
&\Rightarrow \times:\langle 2,1\rangle \Rightarrow 2.
\end{aligned}
$$

**11.3.2 Inner product.** We have seen earlier how this definition works.

**Def** $\text{IP} \equiv (/+)\circ(\alpha\times)\circ\text{trans}$

**11.3.3 Matrix multiply.** This matrix multiplication program yields the product of any pair $\langle m,n\rangle$ of conformable matrices, where each matrix $m$ is represented as the sequence of its rows:

$m = \langle m_1, \ldots, m_r\rangle$
where $m_i = \langle m_{i1}, \ldots, m_{is}\rangle$ for $i = 1, \ldots, r$.

**Def** $\text{MM} \equiv (\alpha\alpha\text{IP})\circ(\alpha\text{distl})\circ\text{distr}\circ[1,\ \text{trans}\circ 2]$

Laws of the algebra of programs

$$[f,g]\circ h \equiv [f\circ h,\ g\circ h]$$

**Proposition**: For all functions $f$, $g$, and $h$ and all objects $x$, $([f,g] \circ h):x \equiv [f \circ h, g \circ h]:x$.

**Proof**:

$([f,g] \circ h):x = [f,g]:(h:x)$
by definition of composition
$= <f:(h:x), g:(h:x)>$
by definition of construction
$= <(f \circ h):x, (g \circ h):x>$
by definition of composition
$= [f \circ h, g \circ h]:x$
by definition of construction □

Proofs of some law

$\text{apndl} \circ [f \circ g, \alpha f \circ h] \equiv \alpha f \circ \text{apndl} \circ [g,h]$

**Proof**. We show that, for every object $x$, both of the above functions yield the same result.

**Case 1**. $h:x$ is neither a sequence nor $\phi$.
Then both sides yield $\perp$ when applied to $x$.

**Case 2**. $h:x = \phi$. Then

$\text{apndl} \circ [f \circ g, \alpha f \circ h]: x$
$= \text{apndl}: <f \circ g:x, \phi> = <f:(g:x)>$
$\alpha f \circ \text{apndl} \circ [g,h]: x$
$= \alpha f \circ \text{apndl}: <g:x, \phi> = \alpha f:<g:x>$
$= <f:(g:x)>$

**Case 3**. $h:x = <y_1, \dots, y_n>$. Then

$\text{apndl} \circ [f \circ g, \alpha f \circ h]: x$
$= \text{apndl}: <f \circ g:x, \alpha f: <y_1, \dots, y_n>>$
$= <f:(g:x), f:y_1, \dots, f:y_n>$
$\alpha f \circ \text{apndl} \circ [g,h]: x$
$= \alpha f \circ \text{apndl}: <g:x, <y_1, \dots, y_n>>$
$= \alpha f:<g:x, y_1, \dots, y_n>$
$= <f:(g:x), f:y_1, \dots, f:y_n>$ □

PROPOSITION 2

Pair & not∘null∘1 →→

$$\text{apndl}\circ[[1^2, 2], \text{distr}\circ[\text{tl}\circ 1, 2]] \equiv \text{distr}$$

where $f$ & $g$ is the function: and∘$[f, g]$, and $f^2 \equiv f\circ f$.

PROOF. We show that both sides produce the same result when applied to any pair $\langle x,y\rangle$, where $x \neq \phi$, as per the stated qualification.

CASE 1. $x$ is an atom or $\perp$. Then distr: $\langle x,y\rangle = \perp$, since $x \neq \phi$. The left side also yields $\perp$ when applied to $\langle x,y\rangle$, since tl∘1:$\langle x,y\rangle = \perp$ and all functions are $\perp$-preserving.

CASE 2. $x = \langle x_1, \ldots, x_n\rangle$. Then

$$
\begin{aligned}
&\text{apndl}\circ[[1^2, 2], \text{distr}\circ[\text{tl}\circ 1, 2]]:\langle x, y\rangle \\
&\quad = \text{apndl}: \langle\langle 1{:}x, y\rangle, \text{distr}: \langle \text{tl}{:}x, y\rangle\rangle \\
&\quad = \text{apndl}: \langle\langle x_1,y\rangle, \phi\rangle = \langle\langle x_1,y\rangle\rangle \quad \text{if tl}{:}x = \phi \\
&\quad = \text{apndl}: \langle\langle x_1,y\rangle, \langle\langle x_2,y\rangle, \ldots, \langle x_n,y\rangle\rangle\rangle \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if tl}{:}x \neq \phi \\
&\qquad = \langle\langle x_1,y\rangle, \ldots, \langle x_n,y\rangle\rangle \\
&\quad = \text{distr}: \langle x,y\rangle
\end{aligned}
$$

□