Probabilistic Model-Building Genetic Algorithms

a.k.a. Estimation of Distribution Algorithms
a.k.a. Iterated Density Estimation Algorithms

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Foreword

- Motivation
  - Genetic and evolutionary computation (GEC) popular.
  - Toy problems great, but difficulties in practice.
  - Must design new representations, operators, tune, …
- This talk
  - Discuss a promising direction in GEC.
  - Combine machine learning and GEC.
  - Create practical and powerful optimizers.

Overview

- Introduction
  - Black-box optimization via probabilistic modeling.
- Probabilistic Model-Building GAs
  - Discrete representation
  - Continuous representation
  - Computer programs (PMBGP)
  - Permutations
- Conclusions

Problem Formulation

- Input
  - How do potential solutions look like?
  - How to evaluate quality of potential solutions?
- Output
  - Best solution (the optimum).
- Important
  - No additional knowledge about the problem.

Why View Problem as Black Box?

- Advantages
  - Separate problem definition from optimizer.
  - Easy to solve new problems.
  - Economy argument.
- Difficulties
  - Almost no prior problem knowledge.
  - Problem specifics must be learned automatically.
  - Noise, multiple objectives, interactive evaluation.

Representations Considered Here

- Start with
  - Solutions are n-bit binary strings.

- Later
  - Real-valued vectors.
  - Program trees.
  - Permutations

Typical Situation

- Previously visited solutions + their evaluation:

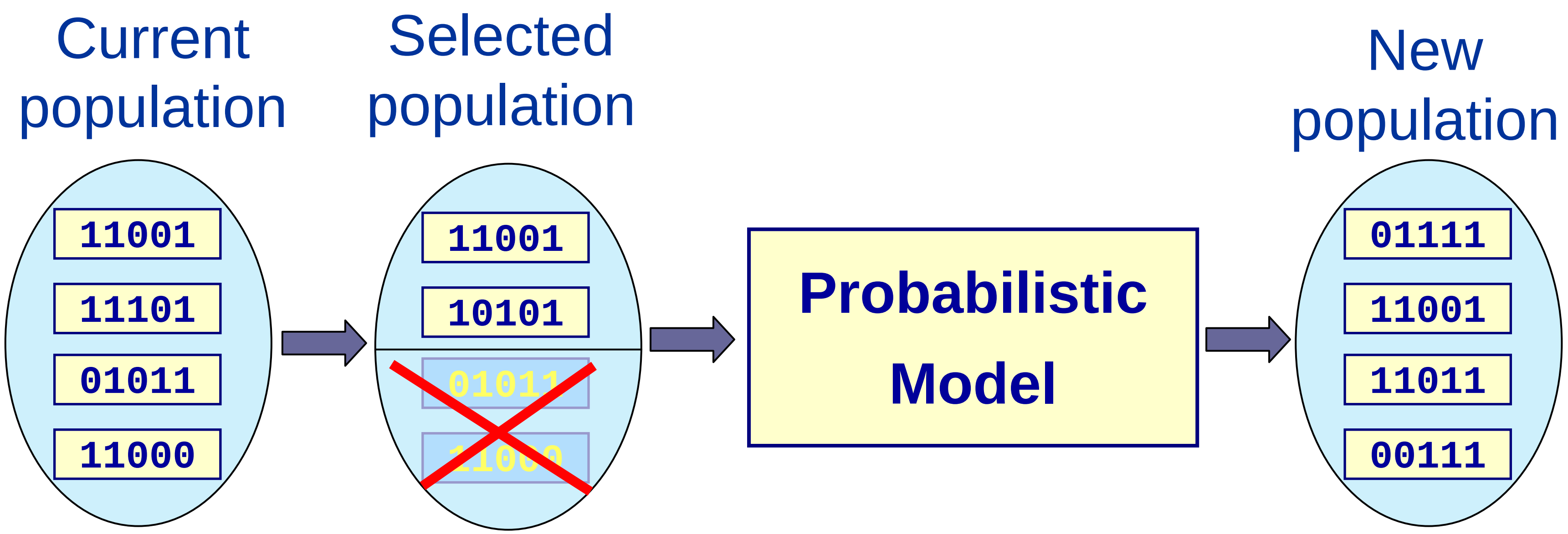
#	Solution	Evaluation
1	00100	1
2	11011	4
3	01101	0
4	10111	3

- Question: What solution to generate next?

Many Answers

- Hill climber
  - Start with a random solution.
  - Flip bit that improves the solution most.
  - Finish when no more improvement possible.
- Simulated annealing
  - Introduce Metropolis.
- Probabilistic model-building GAs
  - Inspiration from GAs and machine learning (ML).

Probabilistic Model-Building GAs



…replace crossover+mutation with learning and sampling probabilistic model

Other Names for PMBGAs

- Estimation of distribution algorithms (EDAs) (Mühlenbein & Paass, 1996)
- Iterated density estimation algorithms (IDEA) (Bosman & Thierens, 2000)

What Models to Use?

- Start with a simple example
  - Probability vector for binary strings.

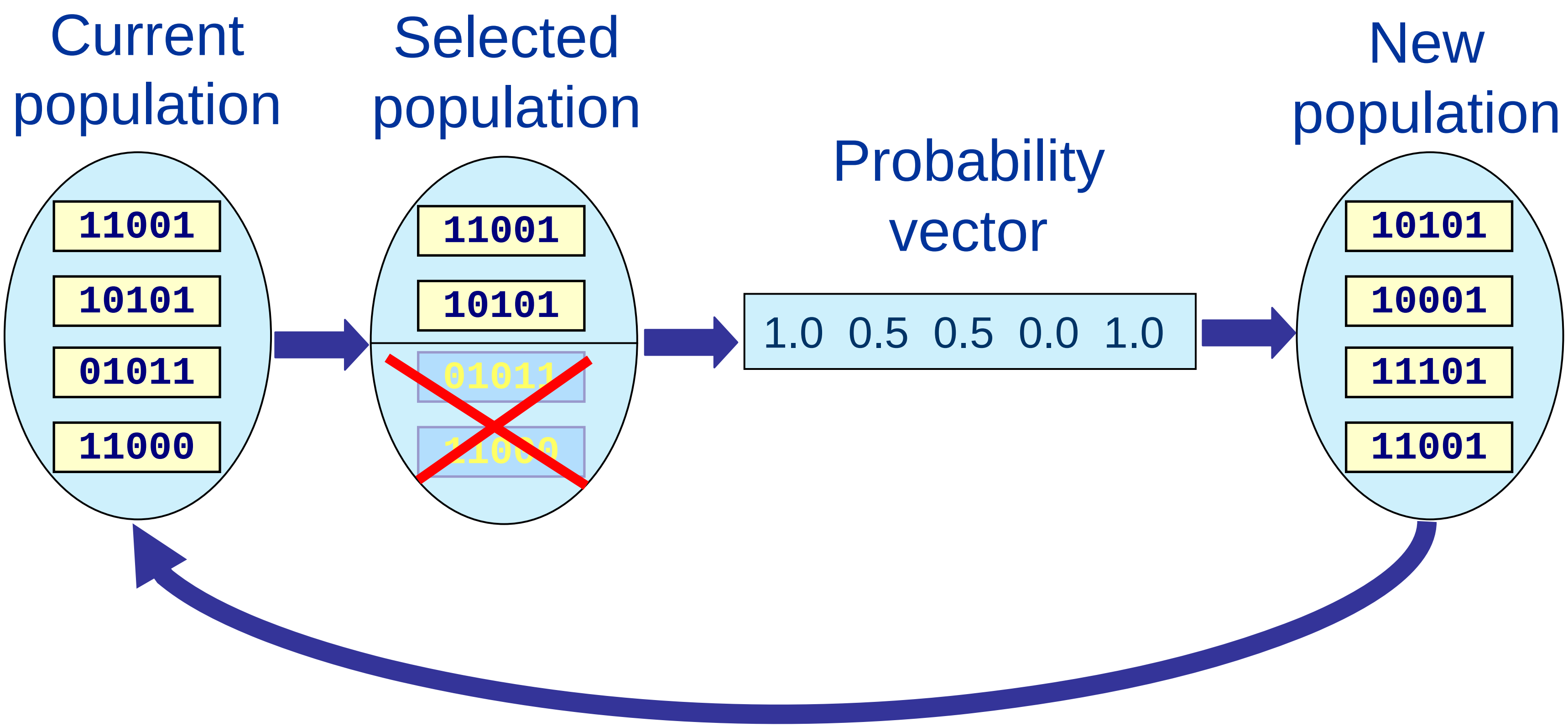
- Later
  - Dependency tree models (COMIT).
  - Bayesian networks (BOA).
  - Bayesian networks with local structures (hBOA).

Probability Vector

- Assume n-bit binary strings.
- Model: Probability vector $p=(p_1, \ldots, p_n)$
  - p_i = probability of 1 in position i
  - Learn p: Compute proportion of 1 in each position.
  - Sample p: Sample 1 in position i with prob. p_i

Example: Probability Vector

(Mühlenbein, Paass, 1996), (Baluja, 1994)



Probability Vector PMBGAs

- PBIL (Baluja, 1995)
  - Incremental updates to the prob. vector.
- Compact GA (Harik, Lobo, Goldberg, 1998)
  - Also incremental updates but better analogy with populations.
- UMDA (Mühlenbein, Paass, 1996)
  - What we showed here.
- DEUM (Shakya et al., 2004)
- All variants perform similarly.

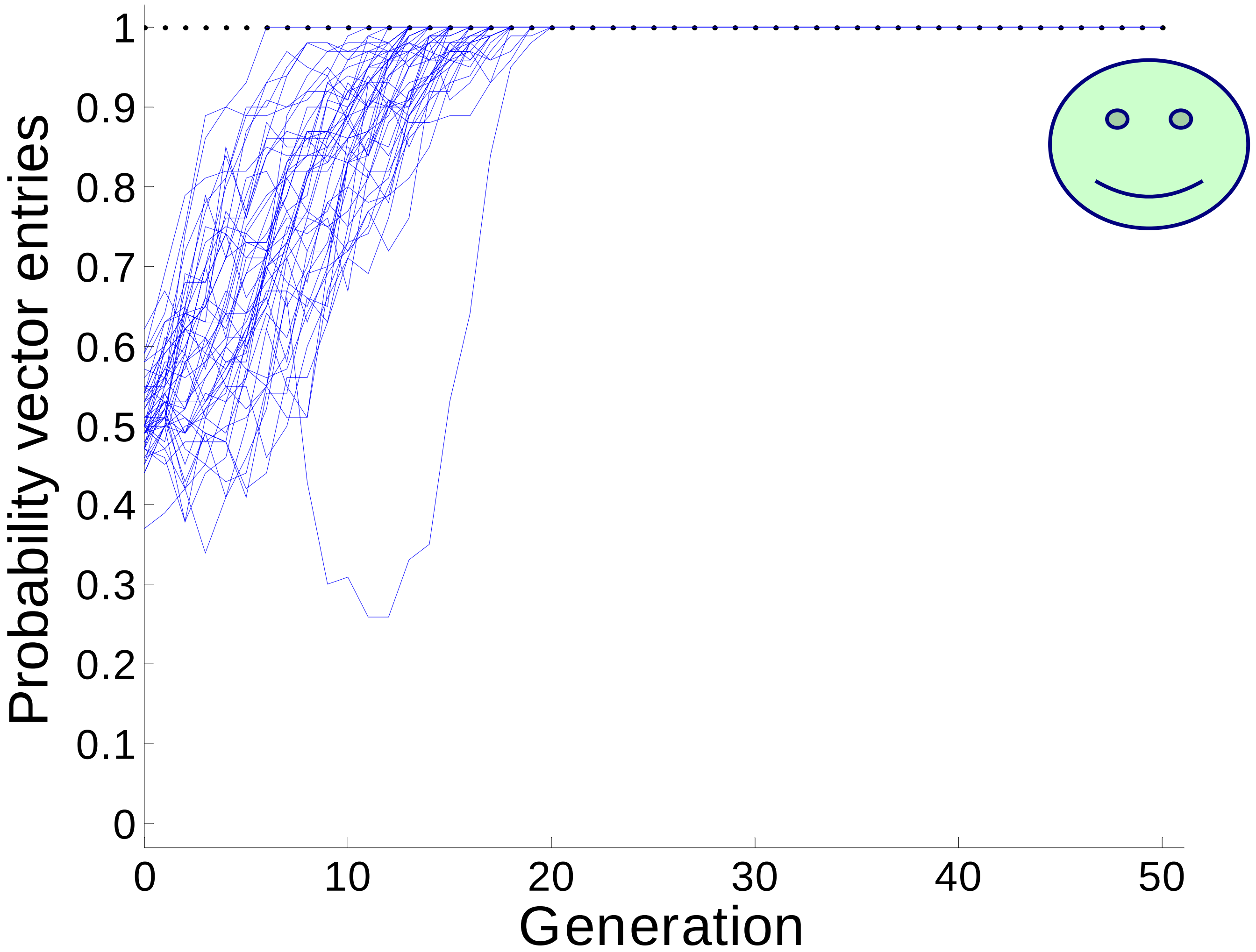
Probability Vector Dynamics

- Bits that perform better get more copies.
- And are combined in new ways.
- But context of each bit is ignored.

- Example problem 1: Onemax

$$f(X_1, X_2, \ldots, X_n) = \sum_{i=1}^{n} X_i$$

Probability Vector on Onemax



Probability Vector: Ideal Scale-up

- O(n log n) evaluations until convergence
  - (Harik, Cantú-Paz, Goldberg, & Miller, 1997)
  - (Mühlenbein, Schlierkamp-Vosen, 1993)
- Other algorithms
  - Hill climber: O(n log n) (Mühlenbein, 1992)
  - GA with uniform: approx. O(n log n)
  - GA with one-point: slightly slower

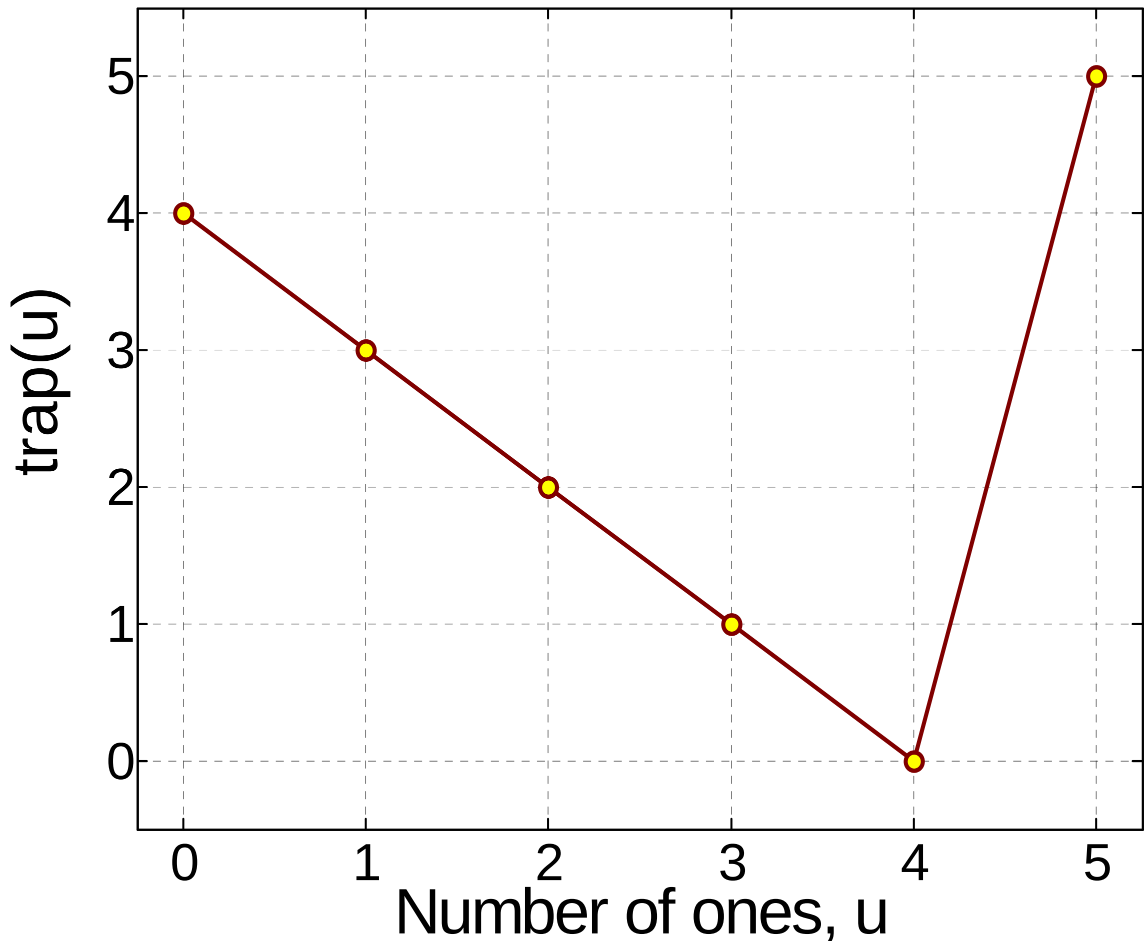
When Does Prob. Vector Fail?

- Example problem 2: Concatenated traps
  - Partition input string into disjoint groups of 5 bits.
  - Groups contribute via trap (ones=number of ones):

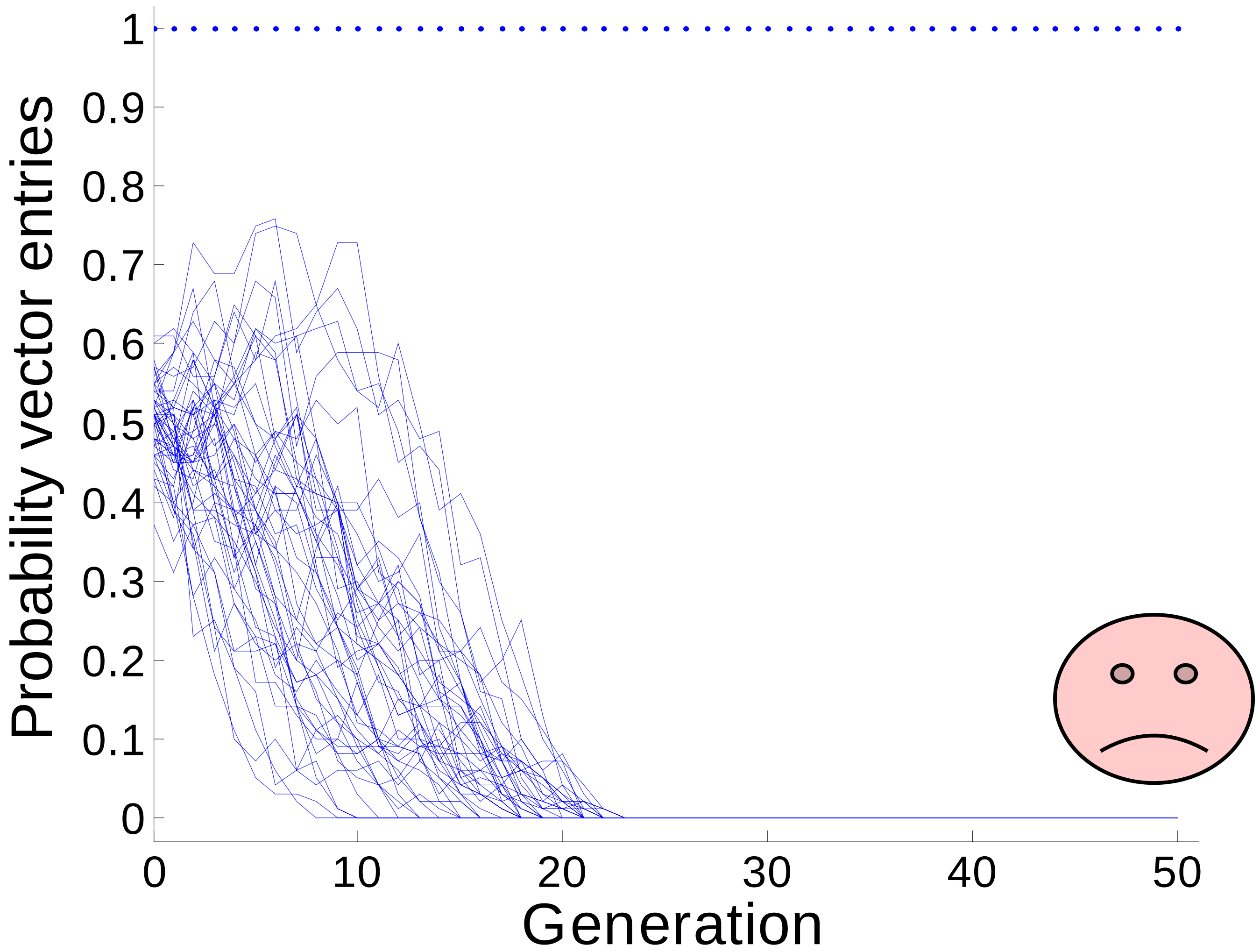
$$trap(ones) = \begin{cases} 5 & \text{if } ones = 5 \\ 4 - ones & \text{otherwise} \end{cases}$$

  - Concatenated trap = sum of single traps
  - Optimum: String 111...1

Trap-5



Probability Vector on Traps



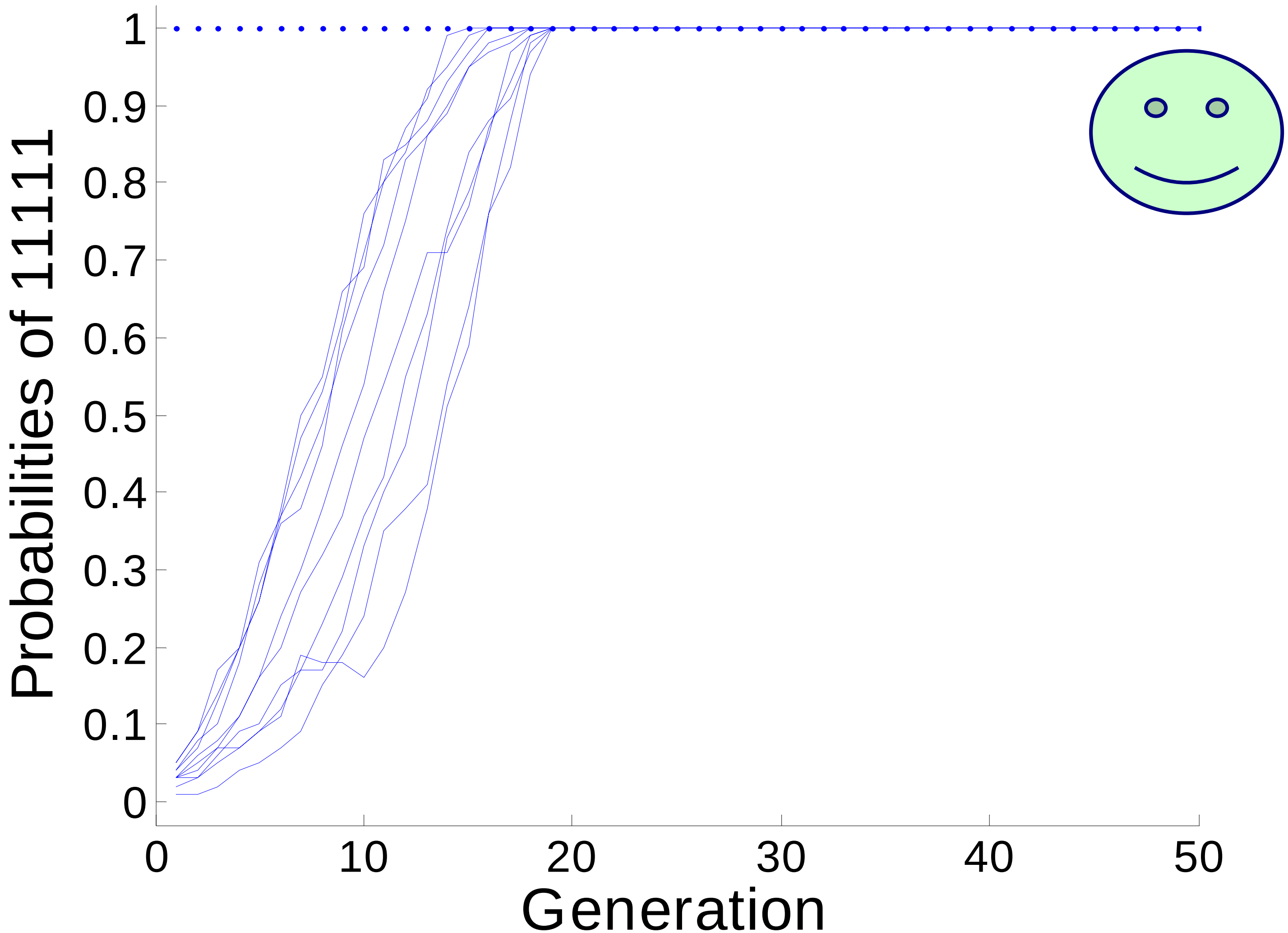
Why Failure?

- Onemax:
  - Optimum in 111...1
  - 1 outperforms 0 on average.
- Traps: optimum in 11111, but
  - f(0****) = 2
  - f(1****) = 1.375
- So single bits are misleading.

How to Fix It?

- Consider 5-bit statistics instead 1-bit ones.
- Then, 11111 would outperform 00000.
- Learn model
  - Compute p(00000), p(00001), ..., p(11111)
- Sample model
  - Sample 5 bits at a time
  - Generate 00000 with p(00000),
    00001 with p(00001), ...

Correct Model on Traps: Dynamics



Good News: Good Stats Work Great!

- Optimum in O(n log n) evaluations.
- Same performance as on onemax!
- Others
  - Hill climber: $O(n^5 \log n)$ = much worse.
  - GA with uniform: $O(2^n)$ = intractable.
  - GA with k-point xover: $O(2^n)$ (w/o tight linkage).

Challenge

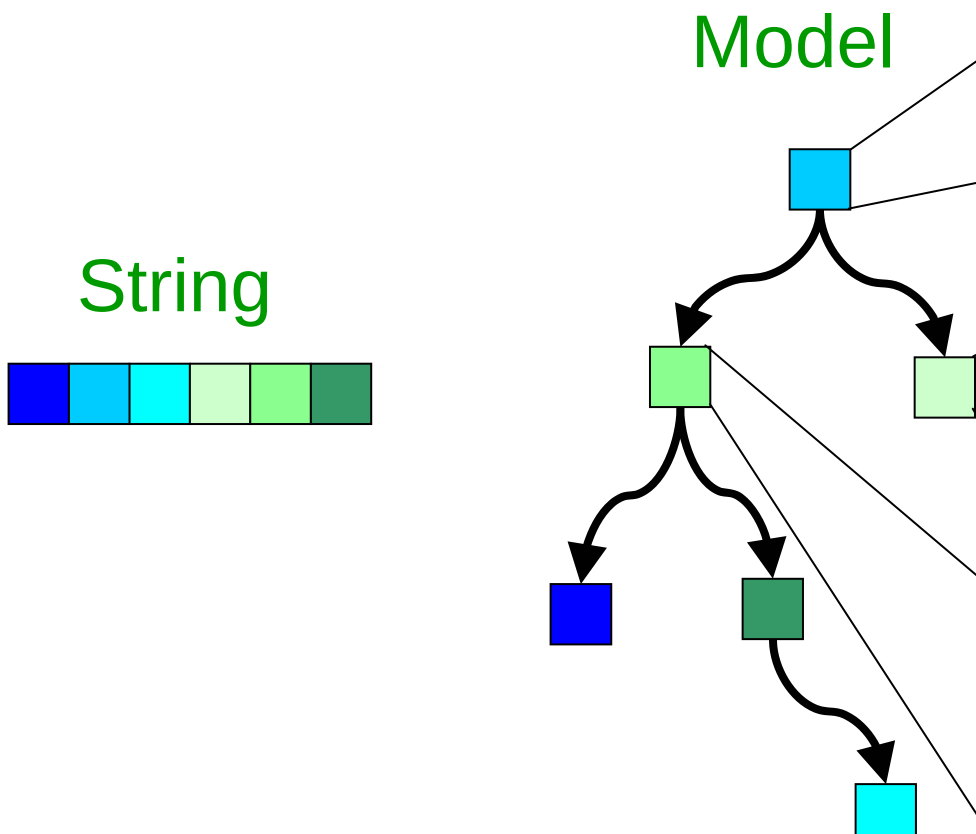
- If we could learn and use relevant context for each position
  - Find non-misleading statistics.
  - Use those statistics as in probability vector.
- Then we could solve problems decomposable into statistics of order at most *k* with at most $O(n^2)$ evaluations!
  - And there are many such problems (Simon, 1968).

What's Next?

- COMIT
  - Use tree models
- Extended compact GA
  - Cluster bits into groups.
- Bayesian optimization algorithm (BOA)
  - Use Bayesian networks (more general).

Beyond single bits: COMIT

(Baluja, Davies, 1997)



P(X=1)
75 %

X	P(Y=1|X)
0	30 %
1	25 %

X	P(Z=1|X)
0	86 %
1	75 %

How to Learn a Tree Model?

- Mutual information:

$$I(X_i, X_j) = \sum_{a,b} P(X_i = a, X_j = b) \log \frac{P(X_i = a, X_j = b)}{P(X_i = a)P(X_j = b)}$$

- Goal
  - Find tree that maximizes mutual information between connected nodes.
  - Will minimize Kullback-Leibler divergence.
- Algorithm
  - Prim's algorithm for maximum spanning trees.

Prim's Algorithm

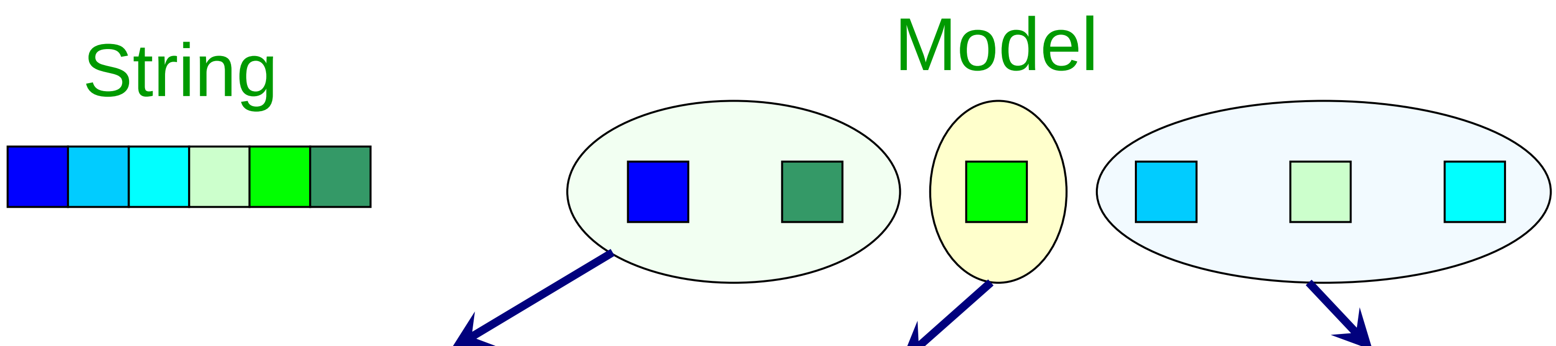
- Start with a graph with no edges.
- Add arbitrary node to the tree.
- Iterate
  - Hang a new node to the current tree.
  - Prefer addition of edges with large mutual information (greedy approach).
- Complexity: $O(n^2)$

Variants of PMBGAs with Tree Models

- COMIT (Baluja, Davies, 1997)
  - Tree models.
- MIMIC (DeBonet, 1996)
  - Chain distributions.
- BMDA (Pelikan, Mühlenbein, 1998)
  - Forest distribution (independent trees or tree)

Beyond Pairwise Dependencies: ECGA

- Extended Compact GA (ECGA) (Harik, 1999).
- Consider groups of string positions.



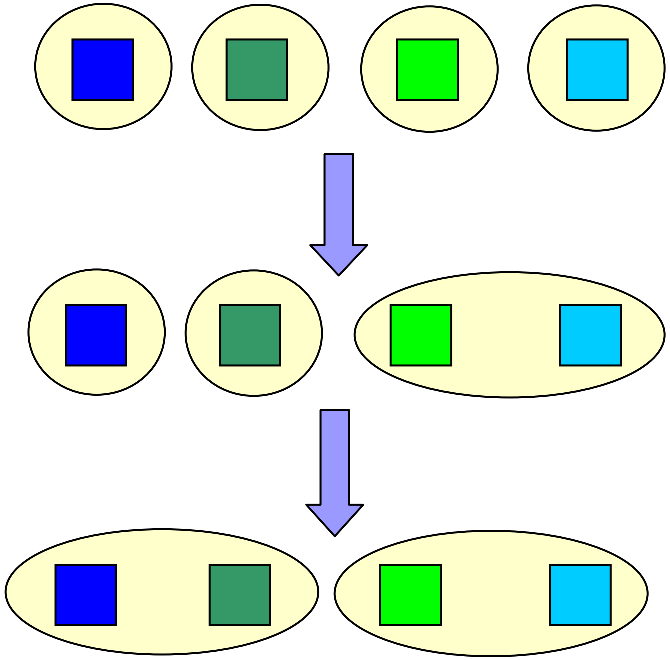
00	16 %
01	45 %
10	35 %
11	4 %

0	86 %
1	14 %

000	17 %
001	2 %
. . .	
111	24 %

Learning the Model in ECGA

- Start with each bit in a separate group.
- Each iteration merges two groups for best improvement.



How to Compute Model Quality?

- ECGA uses minimum description length.
- Minimize number of bits to store model+data:

$$MDL(M,D) = D_{Model} + D_{Data}$$

- Each frequency needs (0.5 log *N*) bits:

$$D_{Model} = \sum_{g \in G} 2^{|g|-1} \log N$$

- Each solution *X* needs -log p(*X*) bits:

$$D_{Data} = -N \sum_{X} p(X) \log p(X)$$

Sampling Model in ECGA

- Sample groups of bits at a time.
- Based on observed probabilities/proportions.
- But can also apply population-based crossover similar to uniform but w.r.t. model.

Building-Block-Wise Mutation in ECGA

- Sastry, Goldberg (2004); Lima et al. (2005)
- Basic idea
  - Use ECGA model builder to identify decomposition
  - Use the best solution for BB-wise mutation
  - For each k-bit partition (building block)
    - Evaluate the remaining 2^{k-1} instantiations of this BB
    - Use the best instantiation of this BB
- Result (for order-k separable problems)
  - BB-wise mutation is $O\left(\sqrt{k}\log n\right)$ times faster than ECGA!
  - But only for separable problems (and similar ones).

What's Next?

- We saw
  - Probability vector (no edges).
  - Tree models (some edges).
  - Marginal product models (groups of variables).

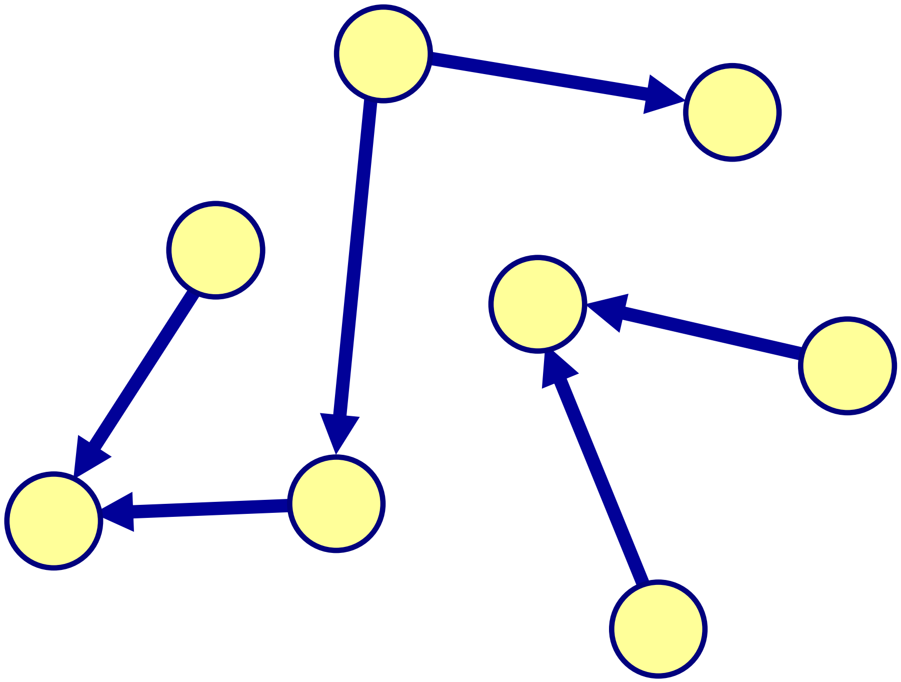
- Next: Bayesian networks
  - Can represent all above and more.

Bayesian Optimization Algorithm (BOA)

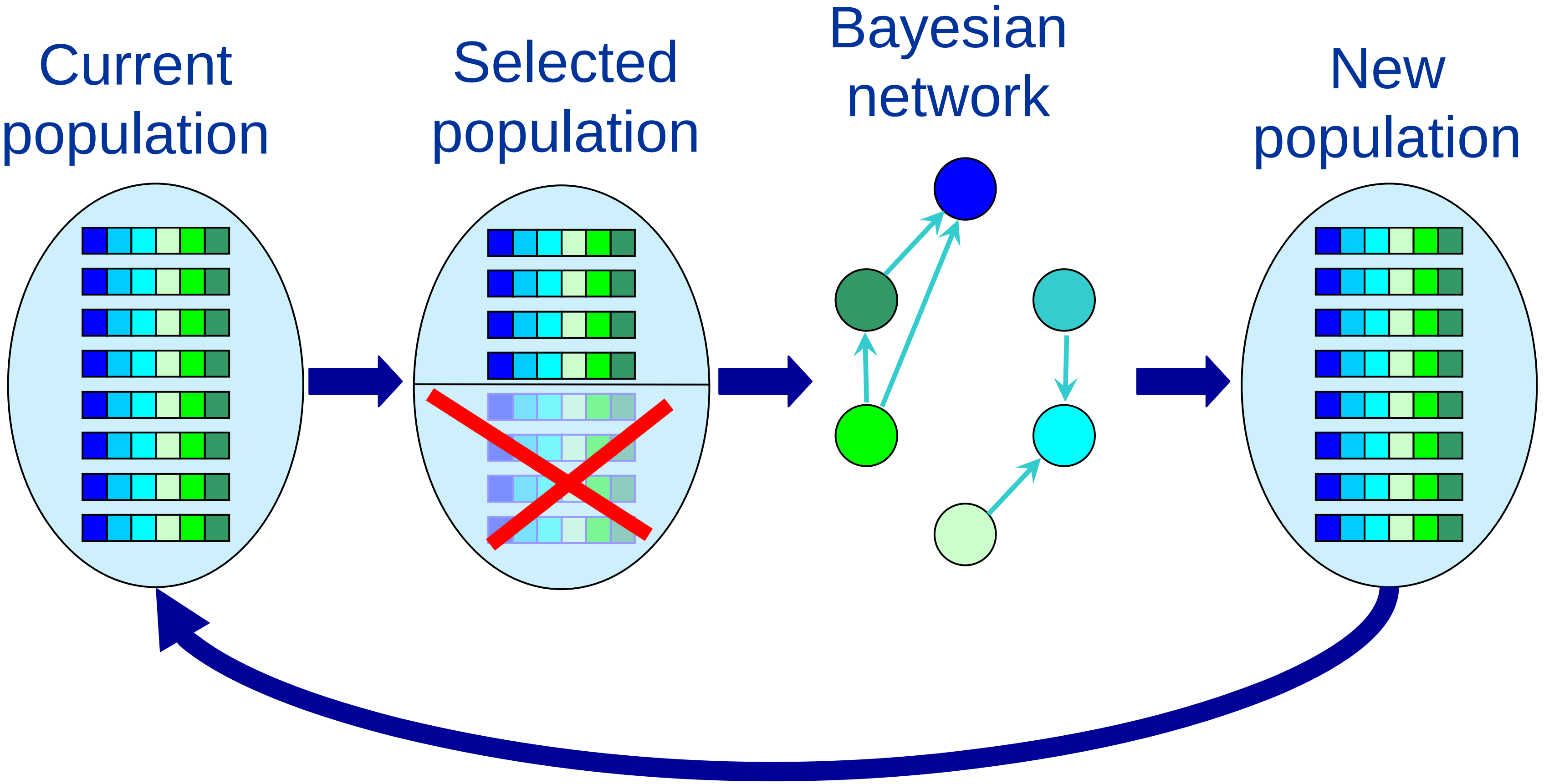
- Pelikan, Goldberg, & Cantú-Paz (1998)
- Use a Bayesian network (BN) as a model.
- Bayesian network
  - Acyclic directed graph.
  - Nodes are variables (string positions).
  - Conditional dependencies (edges).
  - Conditional independencies (implicit).

Example: Bayesian Network (BN)

- Conditional dependencies.
- Conditional independencies.



BOA


Current population
Selected population
Bayesian network
New population

Learning BNs

- Two things again:
  - Scoring metric (as MDL in ECGA).
  - Search procedure (in ECGA done by merging).

Learning BNs: Scoring Metrics

- Bayesian metrics
  - Bayesian-Dirichlet with likelihood equivallence

$$BD(B) = p(B)\prod_{i=1}^{n}\prod_{\pi_i}\frac{\Gamma(m'(\pi_i))}{\Gamma(m'(\pi_i)+m(\pi_i))}\prod_{x_i}\frac{\Gamma(m'(x_i,\pi_i)+m(x_i,\pi_i))}{\Gamma(m'(x_i,\pi_i))}$$

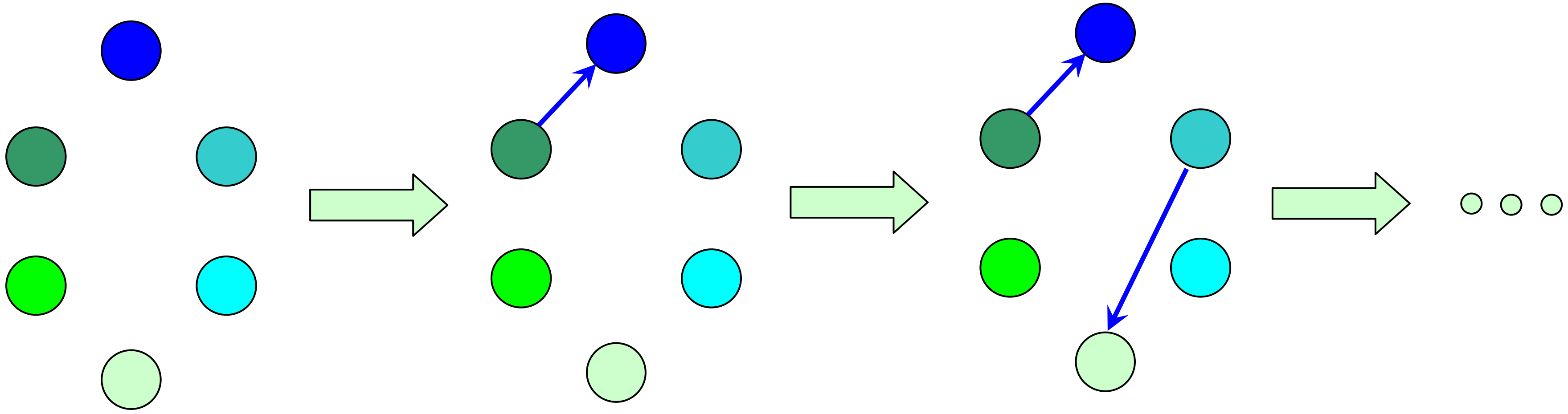
- Minimum description length metrics
  - Bayesian information criterion (BIC)

$$BIC(B) = \sum_{i=1}^{n}\left(-H(X_i\,|\,\Pi_i)N - 2^{|\Pi_i|}\frac{\log_2 N}{2}\right)$$

Learning BNs: Search Procedure

- Start with empty network (like ECGA).
- Execute primitive operator that improves the metric the most (greedy).
- Until no more improvement possible.
- Primitive operators
  - Edge addition (most important).
  - Edge removal.
  - Edge reversal.

Learning BNs: Example



BOA and Problem Decomposition

- Conditions for factoring problem decomposition into a product of prior and conditional probabilities of small order in Mühlenbein, Mahnig, & Rodriguez (1999).
- In practice, approximate factorization sufficient that can be learned automatically.
- Learning makes complete theory intractable.

BOA Theory: Population Sizing

- Initial supply (Goldberg et al., 2001)
  - Have enough stuff to combine. ⇨ $O\left(2^k\right)$
- Decision making (Harik et al, 1997)
  - Decide well between competing partial sols. ⇨ $O\left(\sqrt{n}\log n\right)$
- Drift (Thierens, Goldberg, Pereira, 1998)
  - Don’t lose less salient stuff prematurely. ⇨ $O(n)$
- Model building (Pelikan et al., 2000, 2002)
  - Find a good model. ⇨ $O\left(n^{1.05}\right)$

BOA Theory: Num. of Generations

- Two extreme cases, everything in the middle.
- Uniform scaling
  - Onemax model (Muehlenbein & Schlierkamp-Voosen, 1993)

$$\boxed{O\left(\sqrt{n}\right)}$$

- Exponential scaling
  - Domino convergence (Thierens, Goldberg, Pereira, 1998)

$$\boxed{O(n)}$$

Good News: Challenge Met!

- Theory
  - Population sizing (Pelikan et al., 2000, 2002)
    - Initial supply.
    - Decision making.
    - Drift.
    - Model building.

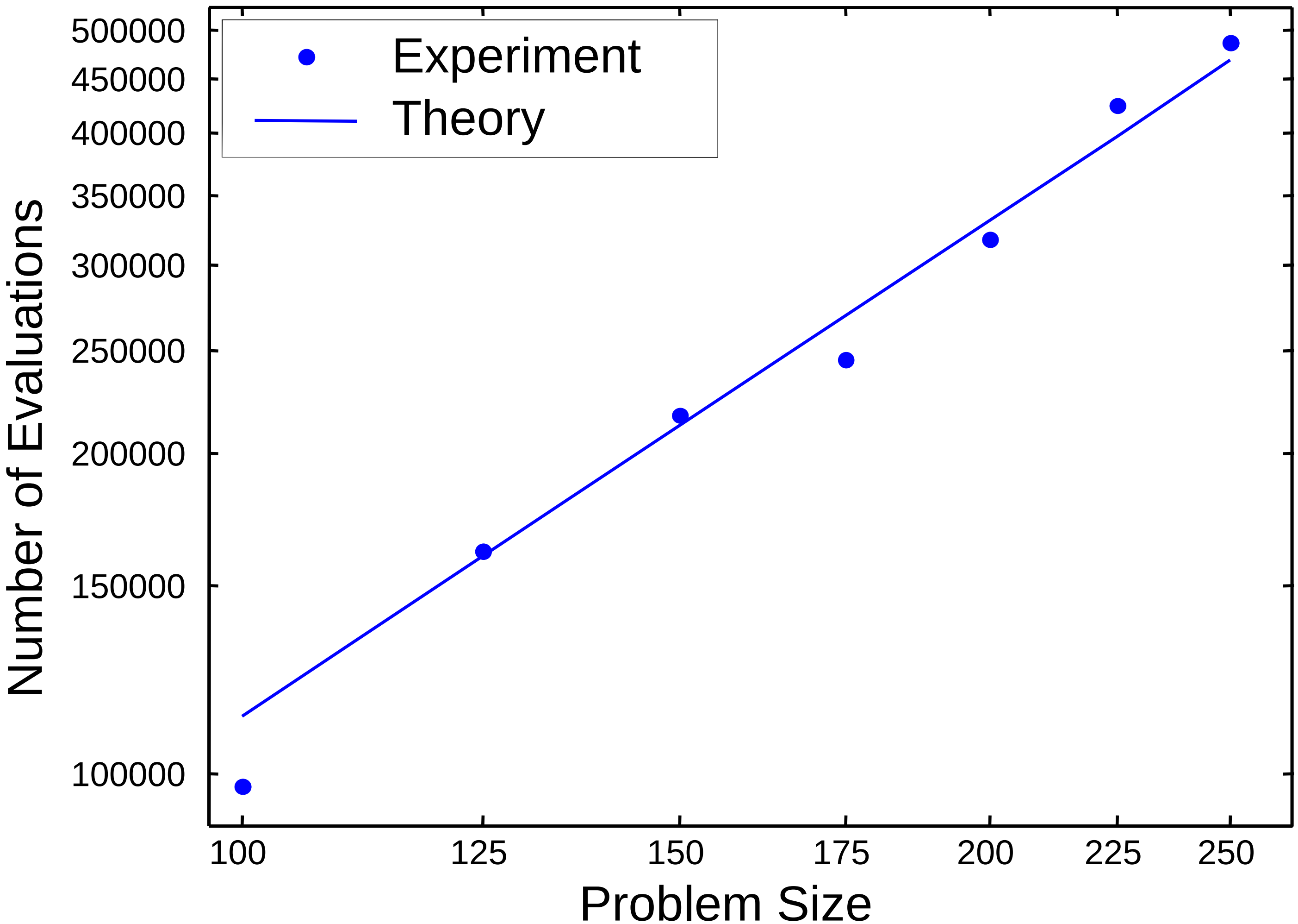
  → $O(n)$ to $O(n^{1.05})$

  - Number of iterations (Pelikan et al., 2000, 2002)
    - Uniform scaling.
    - Exponential scaling.

  → $O(n^{0.5})$ to $O(n)$

- BOA solves order-k decomposable problems in $O(n^{1.55})$ to $O(n^2)$ evaluations!

Theory vs. Experiment (5-bit Traps)

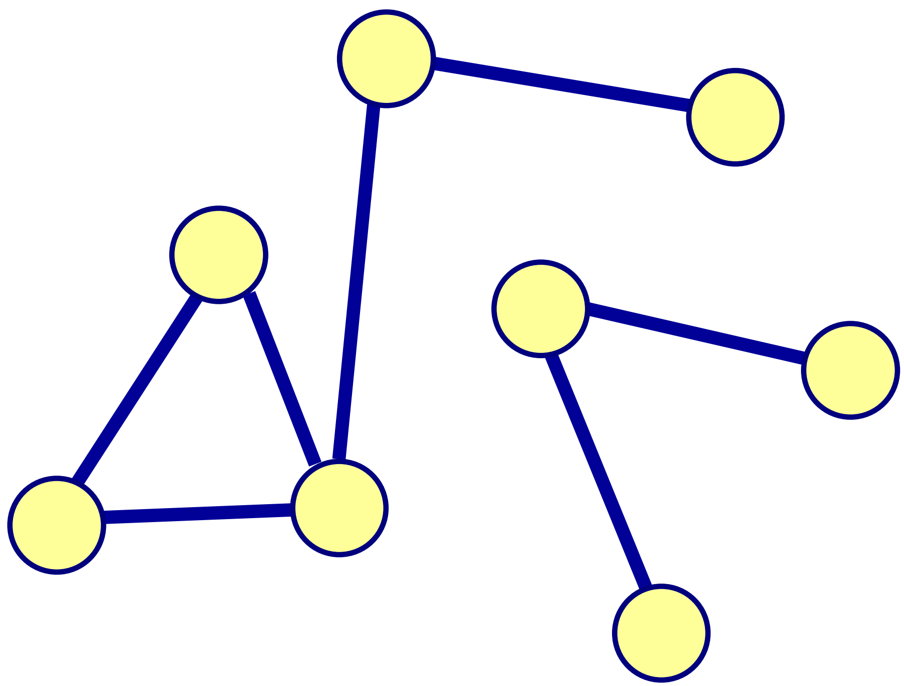


BOA Siblings

- Estimation of Bayesian Networks Algorithm (EBNA) (Etxeberria, Larrañaga, 1999).
- Learning Factorized Distribution Algorithm (LFDA) (Mühlenbein, Mahnig, Rodriguez, 1999).

Another Option: Markov Networks

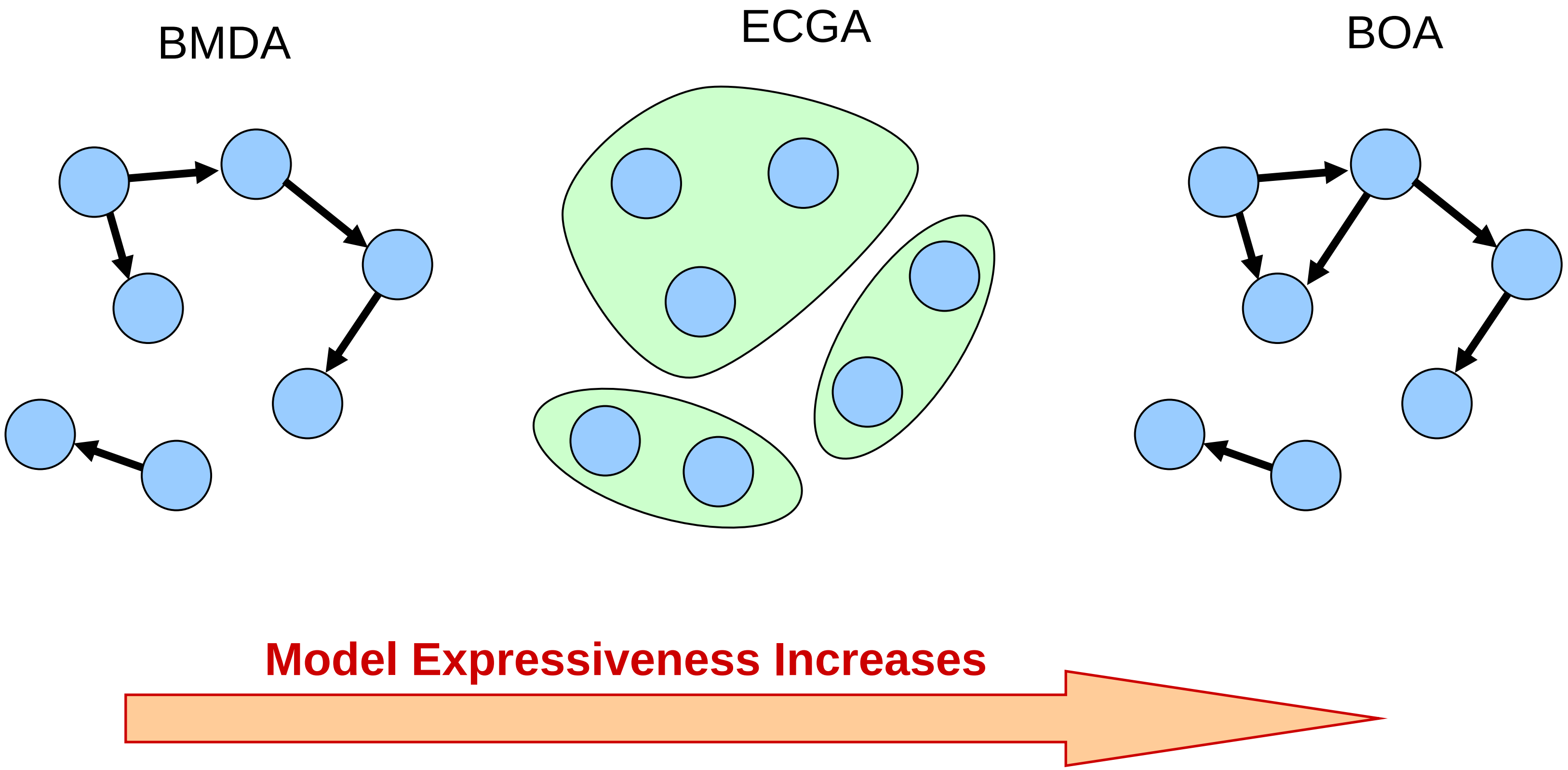
- MN-FDA, MN-EDA (Santana; 2003, 2005)
- Similar to Bayes nets but with undirected edges.



Yet Another Option: Dependency Networks

- Estimation of dependency networks algorithm (EDNA)
  - Gamez, Mateo, Puerta (2007).
  - Use dependency network as a model.
  - Dependency network learned from pairwise interactions.
  - Use Gibbs sampling to generate new solutions.
- Dependency network
  - Parents of a variable= all variables influencing this variable.
  - Dependency network can contain cycles.

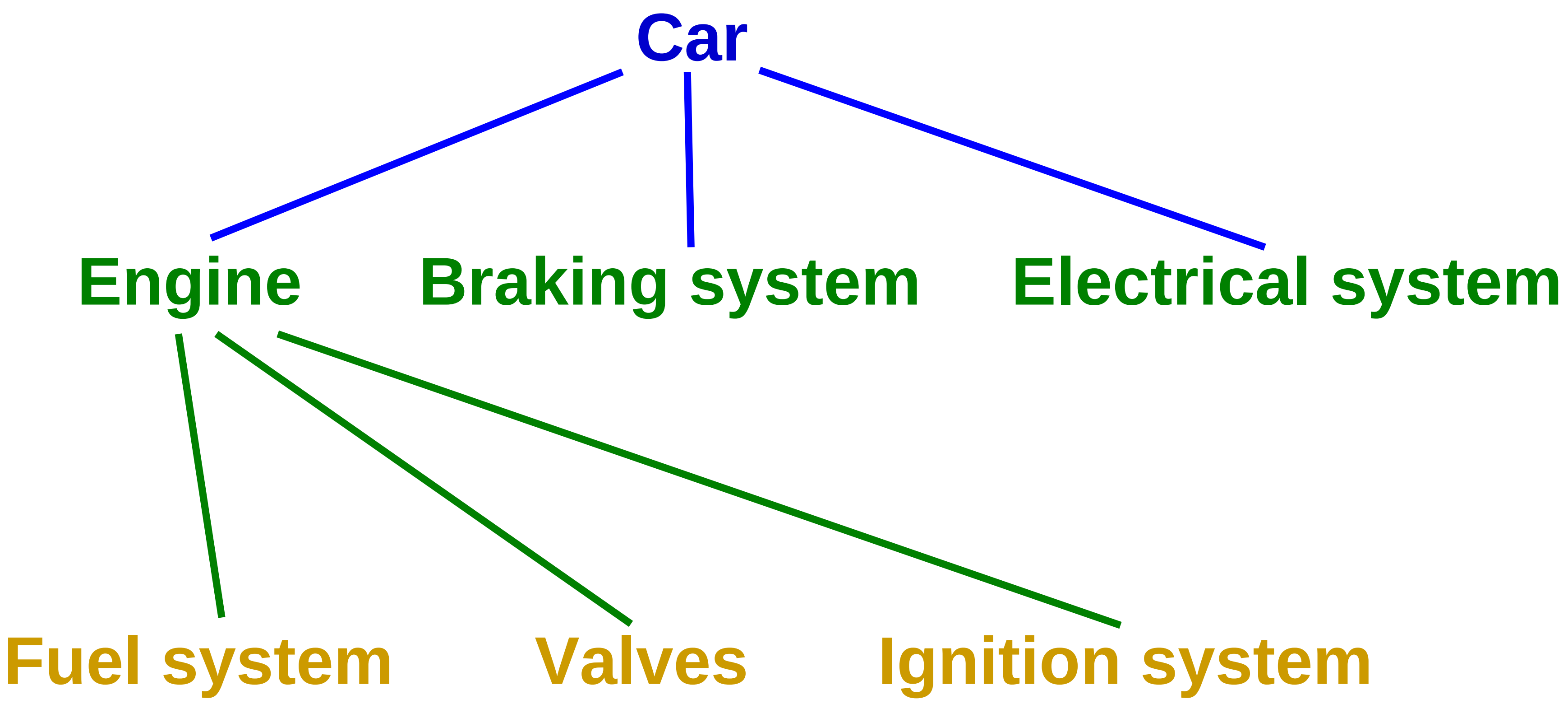
Model Comparison



From single level to hierarchy

- Single-level decomposition powerful.
- But what if single-level decomposition is not enough?
- Learn from humans and nature
  - Decompose problem over multiple levels.
  - Use solutions from lower level as basic building blocks.
  - Solve problem hierarchically.

Hierarchical Decomposition



Three Keys to Hierarchy Success

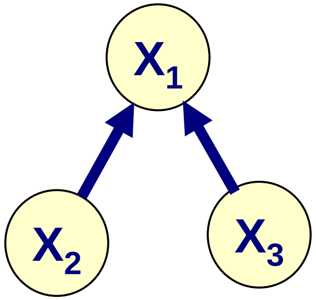
- Proper decomposition
  - Must decompose problem on each level properly.
- Chunking
  - Must represent & manipulate large order solutions.
- Preservation of alternative solutions
  - Must preserve alternative partial solutions (chunks).

Hierarchical BOA (hBOA)

- Pelikan & Goldberg (2000, 2001)
- Proper decomposition
  - Use Bayesian networks like BOA.
- Chunking
  - Use local structures in Bayesian networks.
- Preservation of alternative solutions.
  - Use restricted tournament replacement (RTR).
  - Can use other niching methods.

Local Structures in BNs

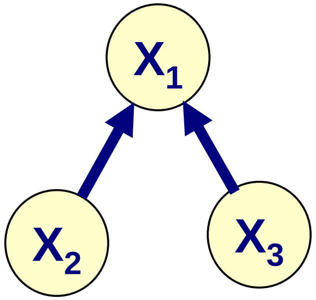
- Look at one conditional dependency.
  - 2^k probabilities for k parents.
- Why not use more powerful representations for conditional probabilities?

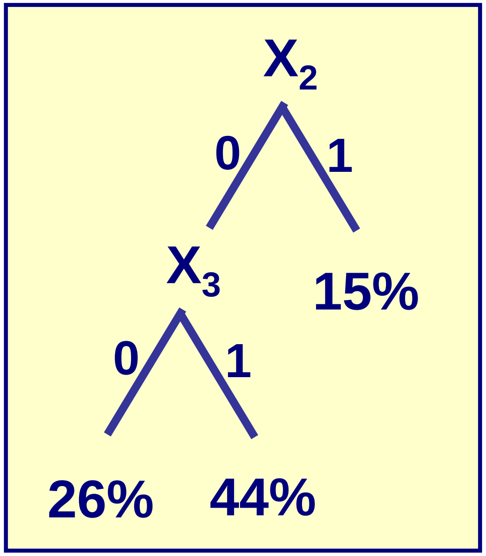


X_2X_3	$P(X_1=0\|X_2X_3)$
00	26 %
01	44 %
10	15 %
11	15 %

Local Structures in BNs

- Look at one conditional dependency.
  - 2^k probabilities for k parents.
- Why not use more powerful representations for conditional probabilities?



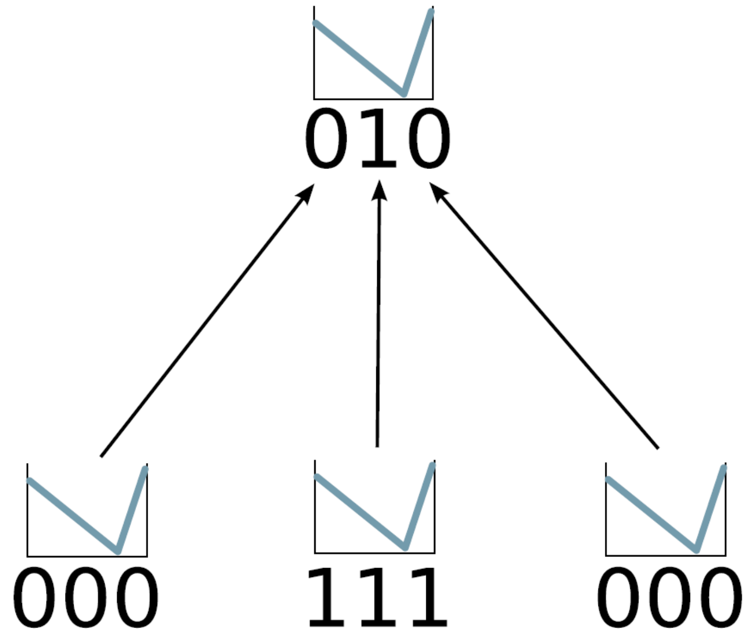


Restricted Tournament Replacement

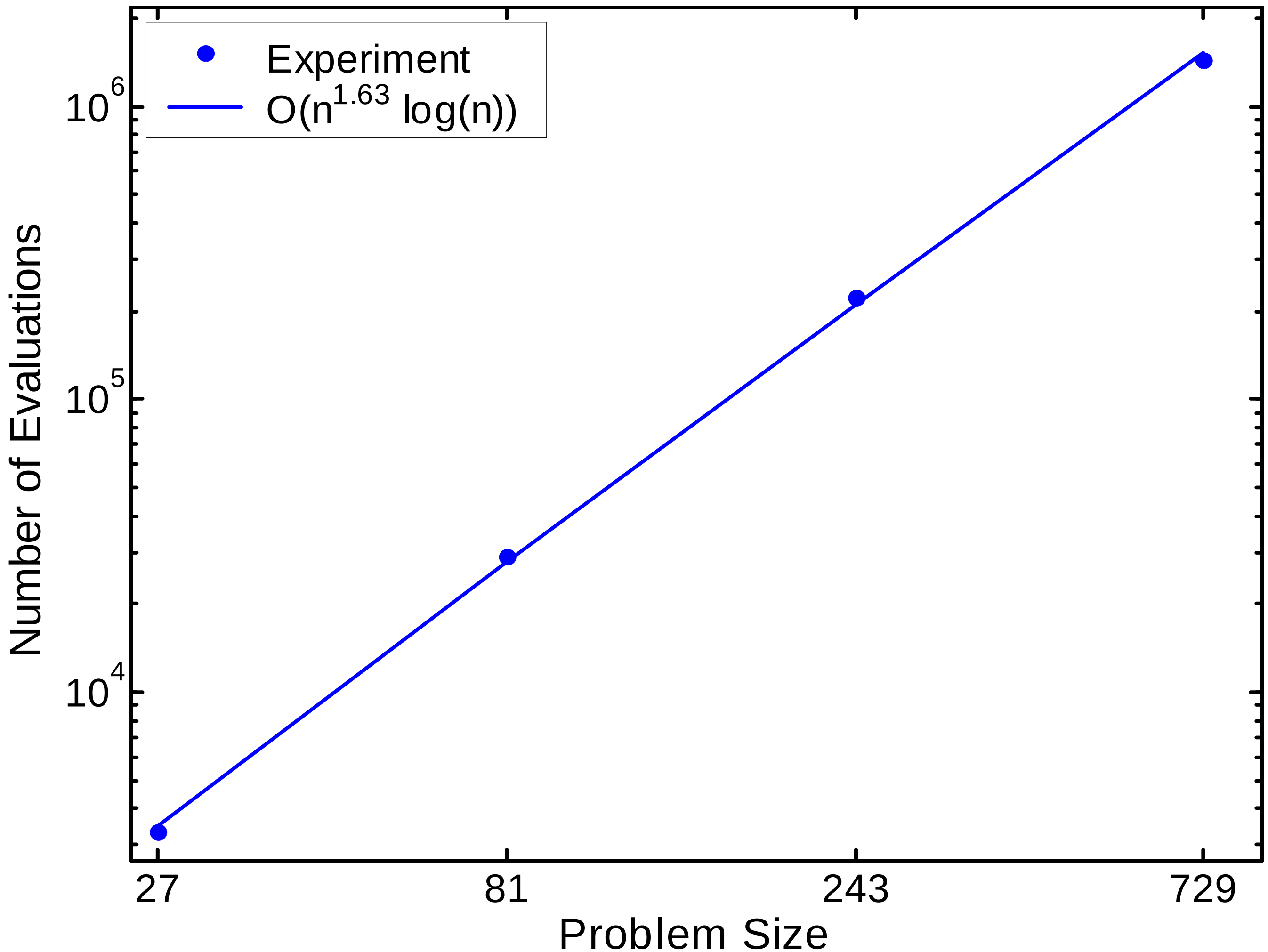
- Used in hBOA for niching.
- Insert each new candidate solution x like this:
  - Pick random subset of original population.
  - Find solution y most similar to x in the subset.
  - Replace y by x if x is better than y.

Hierarchical Traps: The Ultimate Test

- Combine traps on more levels.
- Each level contributes to fitness.
- Groups of bits map to next level.



hBOA on Hierarchical Traps



PMBGAs Are Not Just Optimizers

- PMBGAs provide us with two things
  - Optimum or its approximation.
  - Sequence of probabilistic models.
- Probabilistic models
  - Encode populations of increasing quality.
  - Tell us a lot about the problem at hand.
  - Can we use this information?

Efficiency Enhancement for PMBGAs

- Sometimes $O(n^2)$ is not enough
  - High-dimensional problems (1000s of variables)
  - Expensive evaluation (fitness) function
- Solution
  - Efficiency enhancement techniques

Efficiency Enhancement Types

- 7 efficiency enhancement types for PMBGAs
  - Parallelization
  - Hybridization
  - Time continuation
  - Fitness evaluation relaxation
  - Prior knowledge utilization
  - Incremental and sporadic model building
  - Learning from experience

Multi-objective PMBGAs

- Methods for multi-objective GAs adopted
  - Multi-objective hBOA (from NSGA-II and hBOA)
    (Khan, Goldberg, & Pelikan, 2002)
    (Pelikan, Sastry, & Goldberg, 2005)
  - Another multi-objective BOA (from SPEA2)
    (Laumanns, & Ocenasek, 2002)
  - Multi-objective mixture-based IDEAs
    (Thierens, & Bosman, 2001)
  - Regularity Model Based Multiobjective EDA (RM-MEDA)
    (Zhang, Zhou, Jin, 2008)

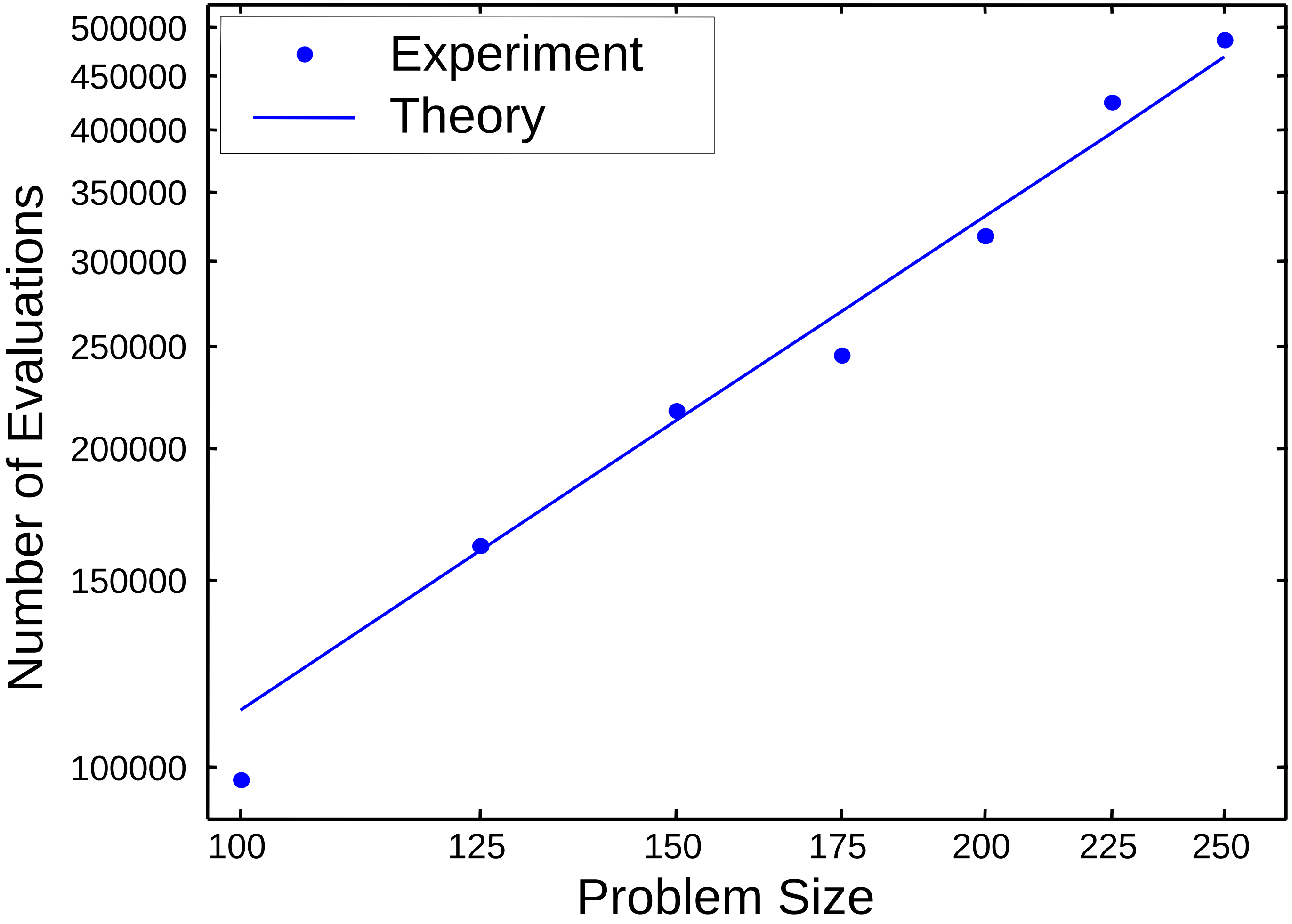
Promising Results with Discrete PMBGAs

- Artificial classes of problems
- Physics
- Bioinformatics
- Computational complexity and AI
- Others

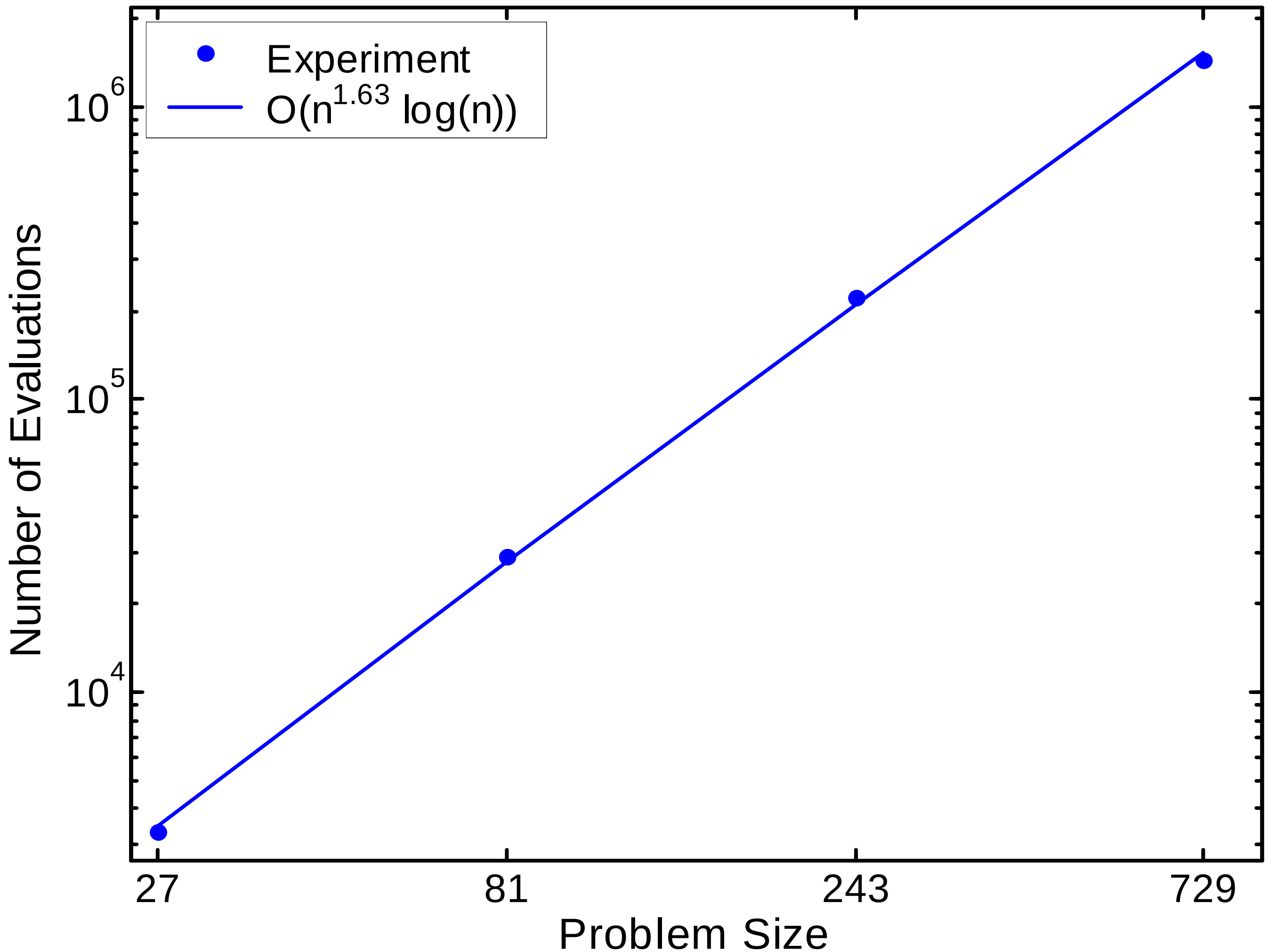
Results: Artificial Problems

- Decomposition
  - Concatenated traps (Pelikan et al., 1998).
- Hierarchical decomposition
  - Hierarchical traps (Pelikan, Goldberg, 2001).
- Other sources of difficulty
  - Exponential scaling, noise (Pelikan, 2002).

BOA on Concatenated 5-bit Traps



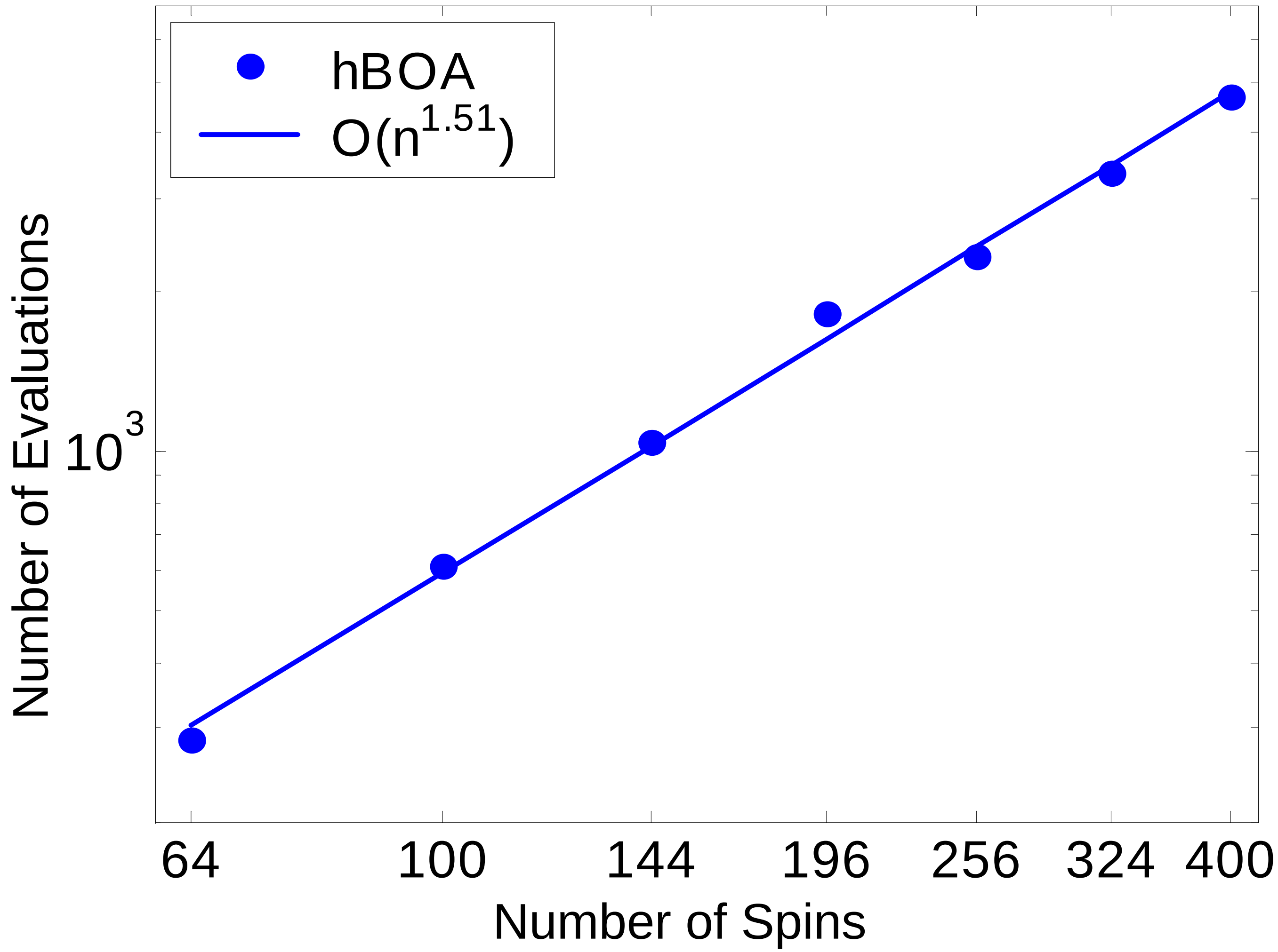
hBOA on Hierarchical Traps



Results: Physics

- Spin glasses (Pelikan et al., 2002, 2006, 2008) (Hoens, 2005) (Santana, 2005) (Shakya et al., 2006)
  - ±J and Gaussian couplings
  - 2D and 3D spin glass
  - Sherrington-Kirkpatrick (SK) spin glass
- Silicon clusters (Sastry, 2001)
  - Gong potential (3-body)

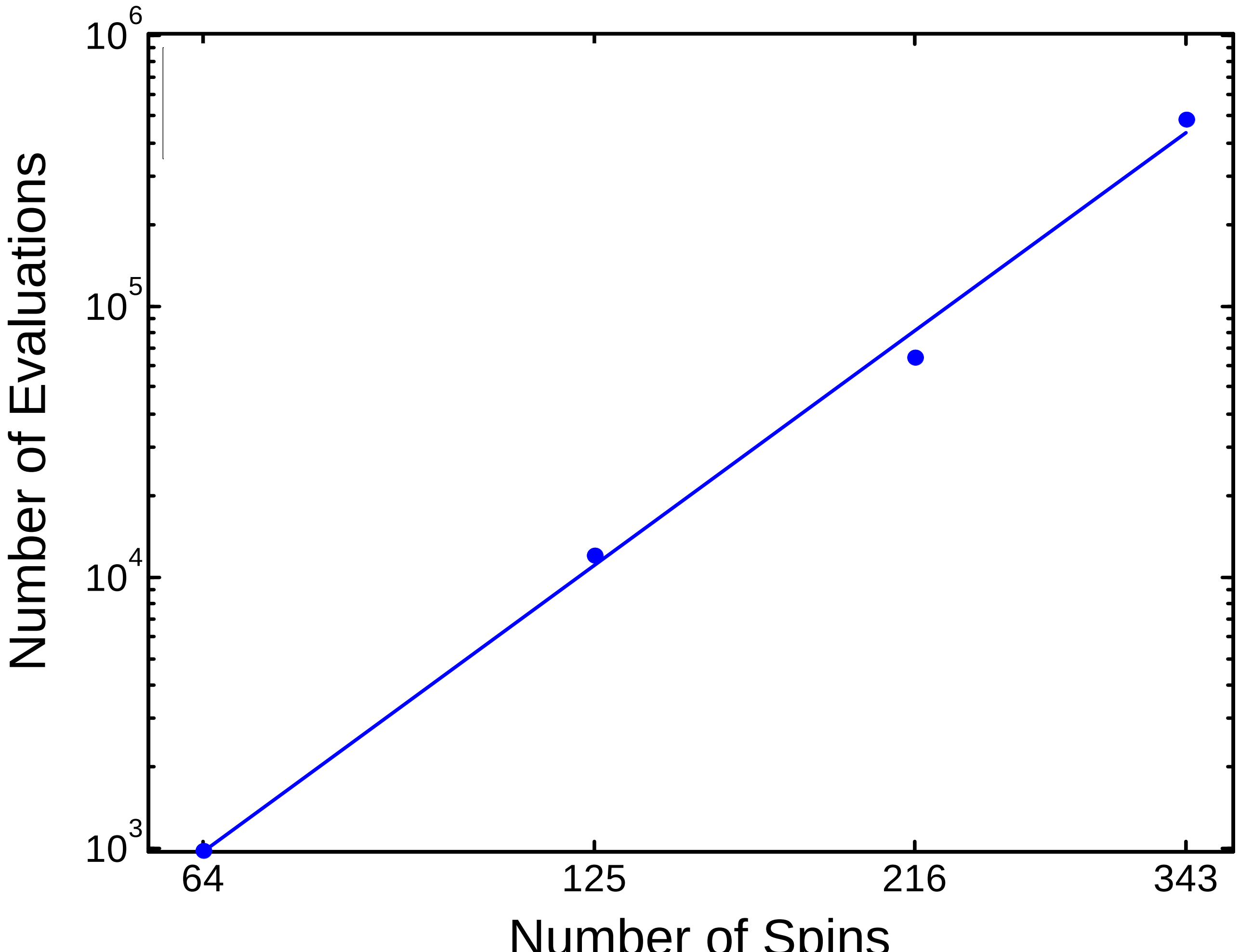
hBOA on Ising Spin Glasses (2D)



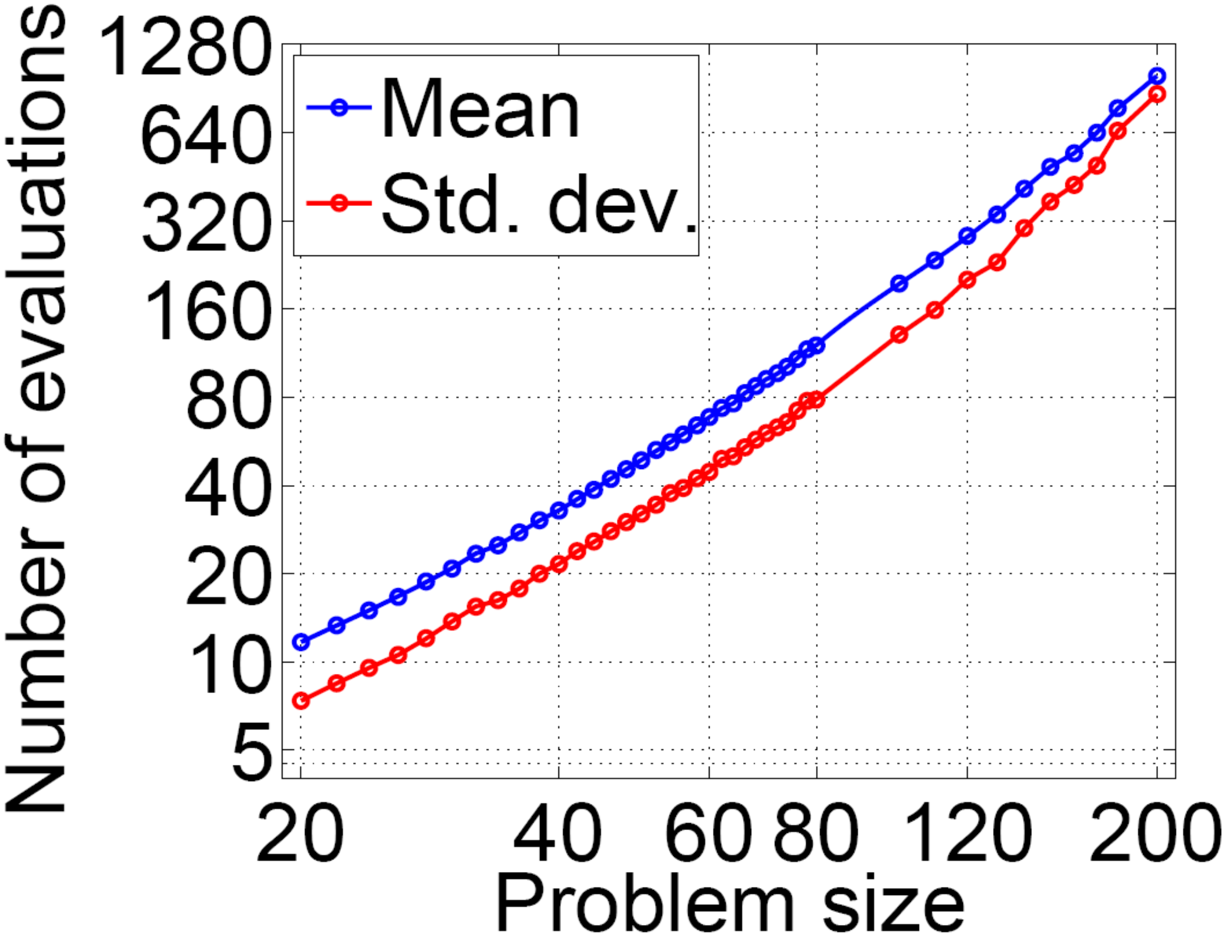
Results on 2D Spin Glasses

- Number of evaluations is O($n^{1.51}$).
- Overall time is O($n^{3.51}$).
- Compare O($n^{3.51}$) to O($n^{3.5}$) for best method (Galluccio & Loebl, 1999)
- Great also on Gaussians.

hBOA on Ising Spin Glasses (3D)



hBOA on SK Spin Glass



Results: Computational Complexity, AI

- MAXSAT, SAT (Pelikan, 2002)
  - Random 3CNF from phase transition.
  - Morphed graph coloring.
  - Conversion from spin glass.

- Feature subset selection (Inza et al., 2001) (Cantu-Paz, 2004)

Results: Some Others

- Military antenna design (Santarelli et al., 2004)
- Groundwater remediation design (Arst et al., 2004)
- Forest management (Ducheyne et al., 2003)
- Nurse scheduling (Li, Aickelin, 2004)
- Telecommunication network design (Rothlauf, 2002)
- Graph partitioning (Ocenasek, Schwarz, 1999; Muehlenbein, Mahnig, 2002; Baluja, 2004)
- Portfolio management (Lipinski, 2005, 2007)
- Quantum excitation chemistry (Sastry et al., 2005)
- Maximum clique (Zhang et al., 2005)
- Cancer chemotherapy optimization (Petrovski et al., 2006)
- Minimum vertex cover (Pelikan et al., 2007)
- Protein folding (Santana et al., 2007)
- Side chain placement (Santana et al., 2007)

Discrete PMBGAs: Summary

- No interactions
  - Univariate models; PBIL, UMDA, cGA.
- Some pairwise interactions
  - Tree models; COMIT, MIMIC, BMDA.
- Multivariate interactions
  - Multivariate models: BOA, EBNA, LFDA.
- Hierarchical decomposition
  - hBOA

Discrete PMBGAs: Recommendations

- Easy problems
  - Use univariate models; PBIL, UMDA, cGA.
- Somewhat difficult problems
  - Use bivariate models; MIMIC, COMIT, BMDA.
- Difficult problems
  - Use multivariate models; BOA, EBNA, LFDA.
- Most difficult problems
  - Use hierarchical decomposition; hBOA.

Real-Valued PMBGAs

- New challenge
  - Infinite domain for each variable.
  - How to model?

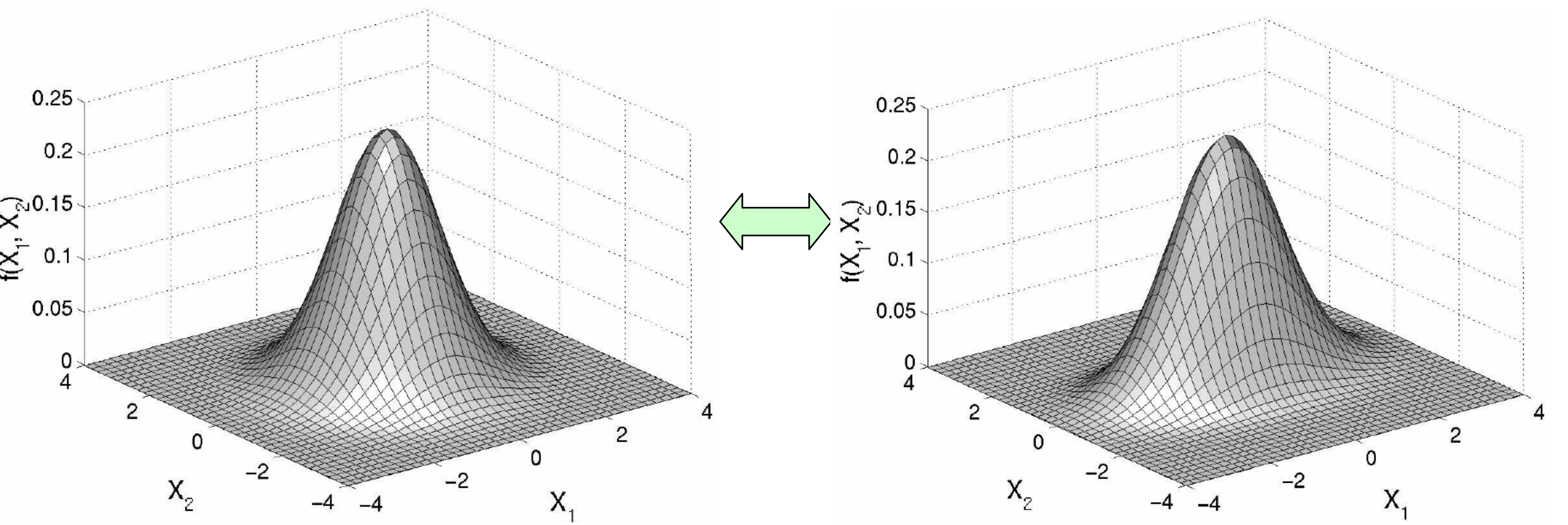
- 2 approaches
  - Discretize and apply discrete model/PMBGA
  - Create model for real-valued variables
    - Estimate pdf.

PBIL Extensions: First Step

- SHCwL: Stochastic hill climbing with learning (Rudlof, Köppen, 1996).
- Model
  - Single-peak Gaussian for each variable.
  - Means evolve based on parents (promising solutions).
  - Deviations equal, decreasing over time.
- Problems
  - No interactions.
  - Single Gaussians=can model only one attractor.
  - Same deviations for each variable.

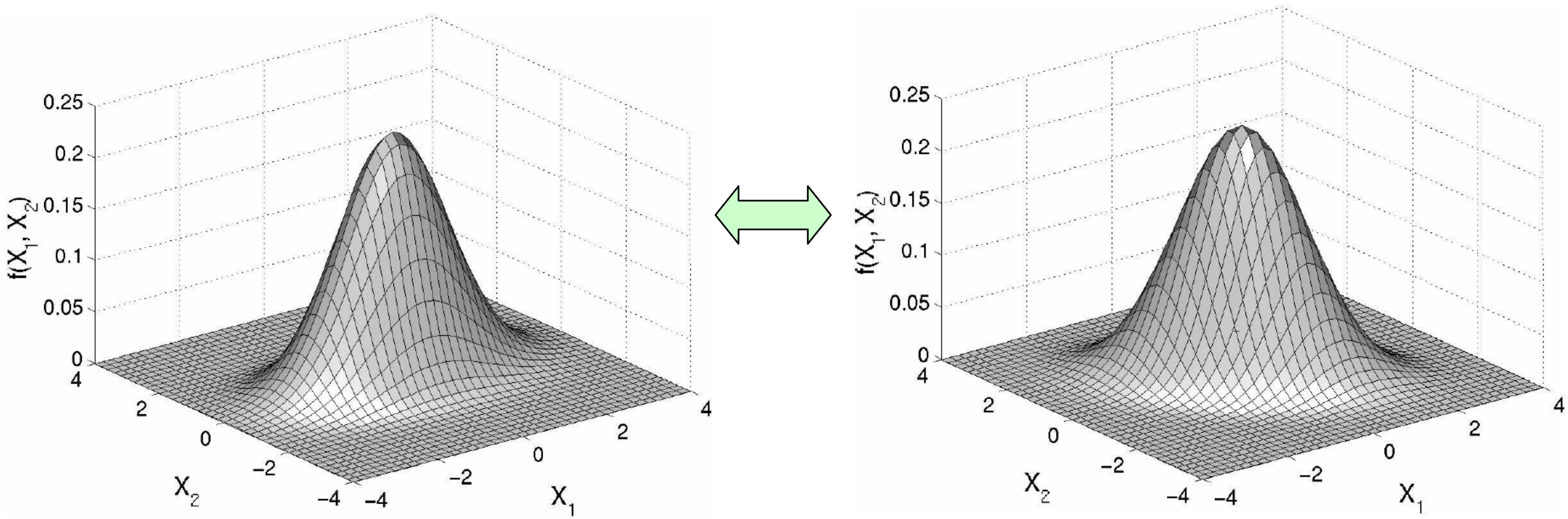
Use Different Deviations

- Sebag, Ducoulombier (1998)
- Some variables have higher variance.
- Use special standard deviation for each



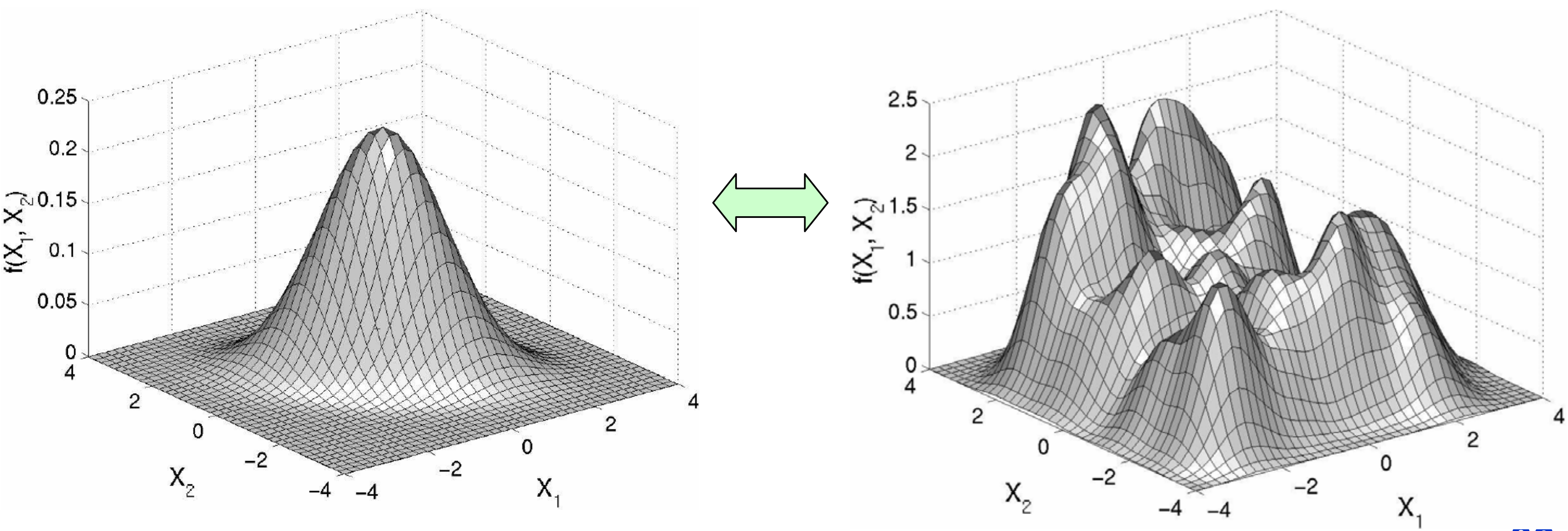
Use Covariance

- Covariance allows rotation of 1-peak Gaussians.
- EGNA (Larrañaga et al., 2000)
- IDEA (Bosman, Thierens, 2000)



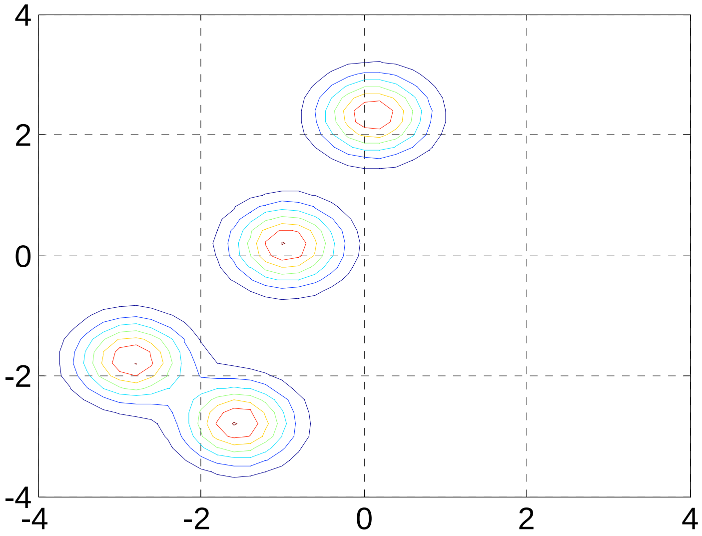
How Many Peaks?

- One Gaussian vs. kernel around each point.
- Kernel distribution similar to ES.
- IDEA (Bosman, Thierens, 2000)



Mixtures: Between One and Many

- Mixture distributions provide transition between one Gaussian and Gaussian kernels.
- Mixture types
  - Over one variable.
    - Gallagher, Frean, & Downs (1999).
  - Over all variables.
    - Pelikan & Goldberg (2000).
    - Bosman & Thierens (2000).
  - Over partitions of variables.
    - Bosman & Thierens (2000).
    - Ahn, Ramakrishna, and Goldberg (2004).



Mixed BOA (mBOA)

- Mixed BOA (Ocenasek, Schwarz, 2002)
- Local distributions
  - A decision tree (DT) for every variable.
  - Internal DT nodes encode tests on other variables
    - Discrete: Equal to a constant
    - Continuous: Less than a constant
  - Discrete variables:
    DT leaves represent probabilities.
  - Continuous variables:
    DT leaves contain a normal kernel distribution.

Real-Coded BOA (rBOA)

- Ahn, Ramakrishna, Goldberg (2003)
- Probabilistic Model
  - Underlying structure: Bayesian network
  - Local distributions: Mixtures of Gaussians
- Also extended to multiobjective problems (Ahn, 2005)

Aggregation Pheromone System (APS)

- Tsutsui (2004)
- Inspired by aggregation pheromones
- Basic idea
  - Good solutions emit aggregation pheromones
  - New candidate solutions based on the density of aggregation pheromones
  - Aggregation pheromone density encodes a mixture distribution

Adaptive Variance Scaling

- Adaptive variance in mBOA
  - Ocenasek et al. (2004)
- Normal IDEAs
  - Bosman et al. (2006, 2007)
  - Correlation-triggered adaptive variance scaling
  - Standard-deviation ratio (SDR) triggered variance scaling

Real-Valued PMBGAs: Discretization

- Idea: Transform into discrete domain.
- Fixed models
  - 2^k equal-width bins with k-bit binary string.
  - Goldberg (1989).
  - Bosman & Thierens (2000); Pelikan et al. (2003).
- Adaptive models
  - Equal-height histograms of 2k bins.
  - k-means clustering on each variable.
  - Pelikan, Goldberg, & Tsutsui (2003); Cantu-Paz (2001).

Real-Valued PMBGAs: Summary

- Discretization
  - Fixed
  - Adaptive
- Real-valued models
  - Single or multiple peaks?
  - Same variance or different variance?
  - Covariance or no covariance?
  - Mixtures?
  - Treat entire vectors, subsets of variables, or single variables?

Real-Valued PMBGAs: Recommendations

- Multimodality?
  - Use multiple peaks.
- Decomposability?
  - All variables, subsets, or single variables.
- Strong linear dependencies?
  - Covariance.
- Partial differentiability?
  - Combine with gradient search.

PMBGP (Genetic Programming)

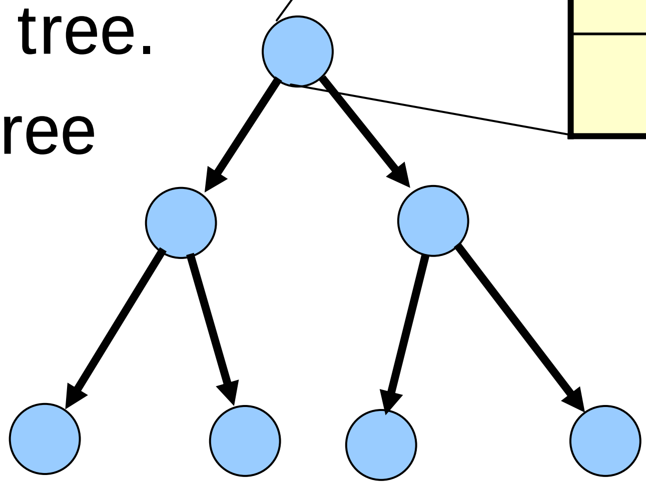
- New challenge
  - Structured, variable length representation.
  - Possibly infinitely many values.
  - Position independence (or not).
  - Low correlation between solution quality and solution structure (Looks, 2006).

- Approaches
  - Use explicit probabilistic models for trees.
  - Use models based on grammars.

PIPE

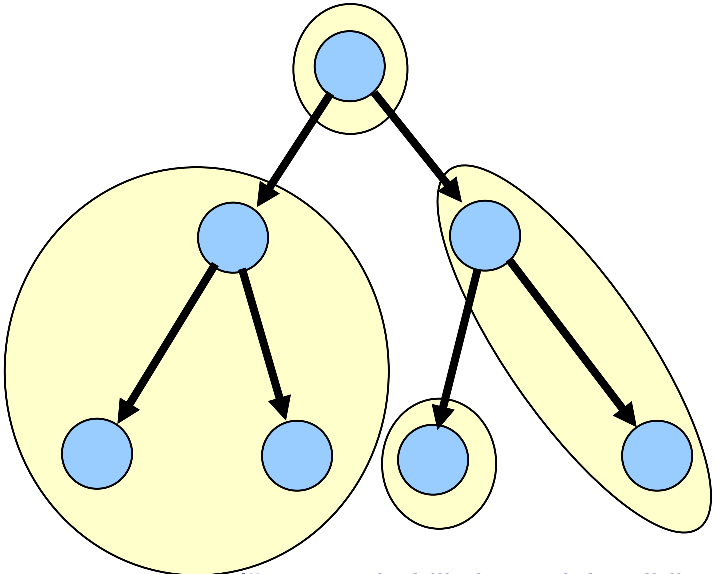
- Probabilistic incremental program evolution (Salustowicz & Schmidhuber, 1997)
- Store frequencies of operators/terminals in nodes of a *maximum* tree.
- Sampling generates tree from top to bottom

X	P(X)
sin	0.15
+	0.35
-	0.35
X	0.15



eCGP

- Sastry & Goldberg (2003)
- ECGA adapted to program trees.
- Maximum tree as in PIPE.
- But nodes partitioned into groups.



BOA for GP

- Looks, Goertzel, & Pennachin (2004)
- Combinatory logic + BOA
  - Trees translated into uniform structures.
  - Labels only in leaves.
  - BOA builds model over symbols in different nodes.
- Complexity build-up
  - Modeling limited to max. sized structure seen.
  - Complexity builds up by special operator.

MOSES

- Looks (2006).
- Evolve demes of programs.
- Each deme represents similar structures.
- Apply PMBGA to each deme (e.g. hBOA).
- Introduce new demes/delete old ones.
- Use normal forms to reduce complexity.

PMBGP with Grammars

- Use grammars/stochastic grammars as models.
- Grammars restrict the class of programs.

- Some representatives
  - Program evolution with explicit learning (Shan et al., 2003)
  - Grammar-based EDA for GP (Bosman, de Jong, 2004)
  - Stochastic grammar GP (Tanev, 2004)
  - Adaptive constrained GP (Janikow, 2004)

PMBGP: Summary

- Interesting starting points available.
- But still lot of work to be done.
- Much to learn from discrete domain, but some completely new challenges.
- Research in progress

PMBGAs for Permutations

- New challenges
  - Relative order
  - Absolute order
  - Permutation constraints
- Two basic approaches
  - Random-key and real-valued PMBGAs
  - Explicit probabilistic models for permutations

Random Keys and PMBGAs

- Bengoetxea et al. (2000); Bosman et al. (2001)
- Random keys (Bean, 1997)
  - Candidate solution = vector of real values
  - Ascending ordering gives a permutation
- Can use any real-valued PMBGA (or GEA)
  - IDEAs (Bosman, Thierens, 2002)
  - EGNA (Larranaga et al., 2001)
- Strengths and weaknesses
  - Good: Can use any real-valued PMBGA.
  - Bad: Redundancy of the encoding.

Direct Modeling of Permutations

- Edge-histogram based sampling algorithm (EHBSA) (Tsutsui, Pelikan, Goldberg, 2003)
  - Permutations of *n* elements
  - Model is a matrix $A=(a_{i,j})_{i,j=1, 2, ..., n}$
  - $a_{i,j}$ represents the probability of edge (i, j)
  - Uses template to reduce exploration
  - Applicable also to scheduling

ICE: Modify Crossover from Model

- ICE
  - Bosman, Thierens (2001).
  - Represent permutations with random keys.
  - Learn multivariate model to factorize the problem.
  - Use the learned model to modify crossover.
- Performance
  - Typically outperforms IDEAs and other PMBGAs that learn and sample random keys.

Multivariate Permutation Models

- Basic approach
  - Use any standard multivariate discrete model.
  - Restrict sampling to permutations in some way.
  - Bengoetxea et al. (2000), Pelikan et al. (2007).
- Strengths and weaknesses
  - Use explicit multivariate models to find regularities.
  - High-order alphabet requires big samples for good models.
  - Sampling can introduce unwanted bias.
  - Inefficient encoding for only relative ordering constraints, which can be encoded simpler.

Conclusions

- Competent PMBGAs exist
  - Scalable solution to broad classes of problems.
  - Solution to previously intractable problems.
  - Algorithms ready for new applications.
- PMBGAs do more than just solve the problem
  - They provide us with sequences of probabilistic models.
  - The probabilistic models tell us a lot about the problem.
- Consequences for practitioners
  - Robust methods with few or no parameters.
  - Capable of learning how to solve problem.
  - But can incorporate prior knowledge as well.
  - Can solve previously intractable problems.

Starting Points

- World wide web
- Books and surveys
  - Larrañaga & Lozano (eds.) (2001). Estimation of distribution algorithms: A new tool for evolutionary computation. Kluwer.
  - Pelikan et al. (2002). A survey to optimization by building and using probabilistic models. Computational optimization and applications, 21(1), pp. 5-20.
  - Pelikan (2005). Hierarchical BOA: Towards a New Generation of Evolutionary Algorithms. Springer.
  - Lozano, Larrañaga, Inza, Bengoetxea (2007). Towards a New Evolutionary Computation: Advances on Estimation of Distribution Algorithms, Springer.
  - Pelikan, Sastry, Cantu-Paz (eds.) (2007). Scalable Optimization via Probabilistic Modeling: From Algorithms to Applications, Springer.

Online Code (1/2)

- BOA, BOA with decision graphs, dependency-tree EDA
  http://medal.cs.umsl.edu/
- ECGA, xi-ary ECGA, BOA, and BOA with decision trees/graphs
  http://www-illigal.ge.uiuc.edu/
- mBOA
  http://jiri.ocenasek.com/
- PIPE
  http://www.idsia.ch/~rafal/
- Real-coded BOA
  http://www.evolution.re.kr/

Online Code (2/2)

- Demos of APS and EHBSA
  http://www.hannan-u.ac.jp/~tsutsui/research-e.html

- RM-MEDA: A Regularity Model Based Multiobjective EDA
  Differential Evolution + EDA hybrid
  http://cswww.essex.ac.uk/staff/qzhang/mypublication.htm

- Naive Multi-objective Mixture-based IDEA (MIDEA)
  Normal IDEA-Induced Chromosome Elements Exchanger (ICE)
  Normal Iterated Density-Estimation Evolutionary Algorithm (IDEA)
  http://homepages.cwi.nl/~bosman/code.html