# *Introduction to Quantum Computing*

by

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# Overview

From bits to qubits: Dirac notation, density matrices, measurement, Bloch sphere

Quantum circuits: basic single-qubit & two-qubit gates, multipartite quantum states
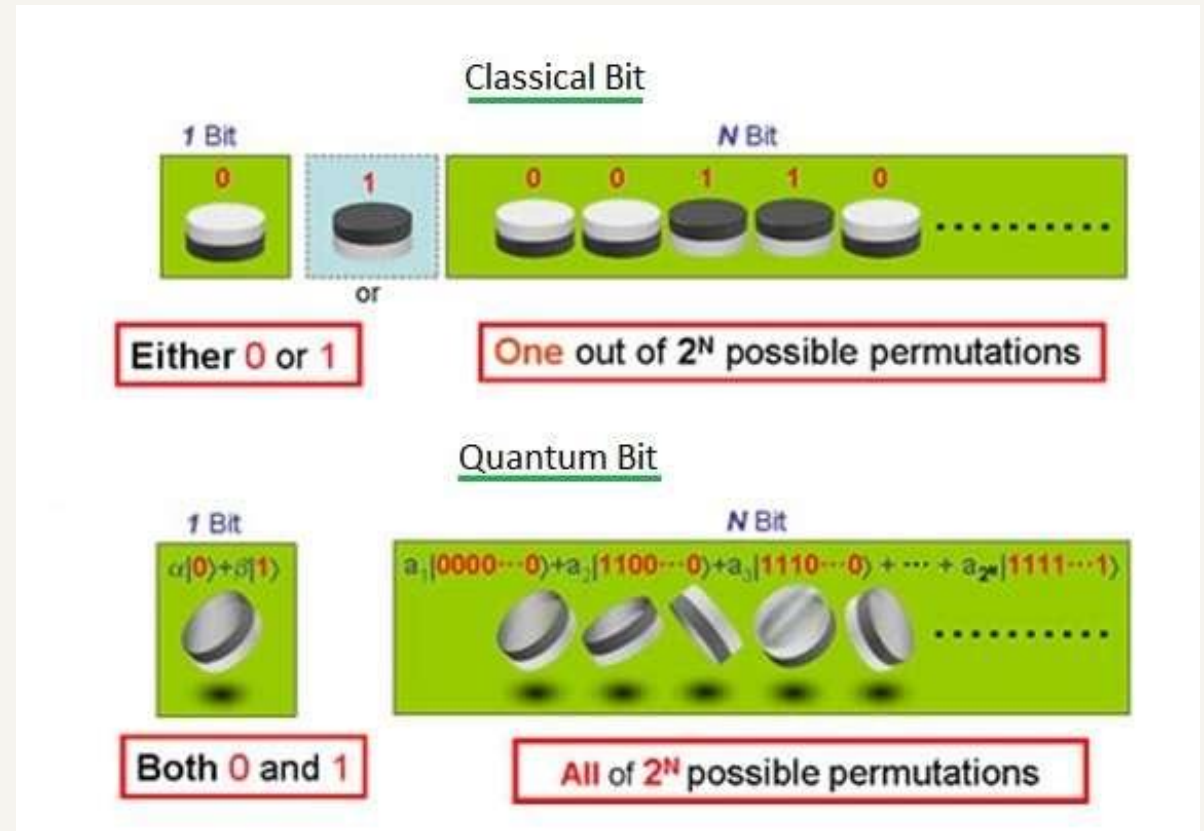
Entanglement: Bell states, Teleportation, Superdense coding

Quantum algorithms: Deutsch-Jozsa algorithm, Grover's algorithm

# *From bits to qubits*



- Superpositions allow to perform calculations on many states at the same time.
  - ➤ Quantum algorithms with **exponential speed-up**.
- But: Once we measure the superposition state, it collapse to one of its states.
- We can use **interference effects** to keep the right answer.

Photo courtesy of : https://medium.com/qntm/qntm-entering-the-era-of-quantum-computing

# Dirac notation & density matrices

- It used to describe quantum states: Let a, b are 2-dimensional vector with complex entries.

  ➢ ket: $|a\rangle = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$

  ➢ bra: $\langle b| = |b\rangle^+ = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}^+ = (b_0^* \; b_1^*)$

  ➢ bra-ket: $\langle b|a\rangle = a_0 b_0{}^* + a_1 b_1{}^* = \langle a|b\rangle^* \in \mathbb{C}$ (inner product)

  ➢ ket-bra: $|a\rangle\langle b| = \begin{pmatrix} a_0 b_0{}^* & a_0 b_1{}^* \\ a_1 b_0{}^* & a_1 b_1{}^* \end{pmatrix}$ (2x2 matrix)

# *Dirac notation & density matrices*

- The pure states are $|0\rangle = \binom{1}{0}, |1\rangle = \binom{0}{1}$, which are orthogonal: $\langle 0|1\rangle = 1.0 + 0.1 = 0$

- $|0\rangle\langle 0| = \binom{1}{0}(1\ 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, |1\rangle\langle 1| = \binom{0}{1}(0\ 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

- $P = \begin{pmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{pmatrix} = P_{00}|0\rangle\langle 0| + P_{01}|0\rangle\langle 1| + P_{10}|1\rangle\langle 0| + P_{11}|1\rangle\langle 1|$

- All quantum states can be described by density matrices.

- All quantum states are normalized, i.e., $\langle\psi|\psi\rangle = 1$, e.g., $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

- A density matrix is pure, if $P = |\psi\rangle\langle\psi|$, otherwise it is mixed.
  - $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 0| \rightarrow$ Pure, $P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = |1\rangle\langle 1| \rightarrow$ Pure
  - $P = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \rightarrow$ Mixed
  - $P = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}(|0\rangle - |1\rangle)(|0\rangle - |1\rangle) \rightarrow$ Pure

# *Measurement*
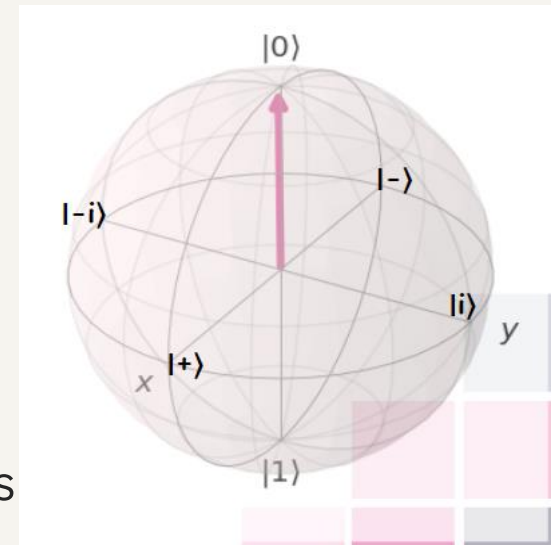


- We choose orthogonal base to describe and measure quantum states (projective measurement).

- During a measurement onto the basis $\{|0\rangle, |1\rangle\}$, the states will collapse into either state $|0\rangle$ or $|1\rangle$, as those are the eigenstates of $\sigma_Z$, we call this a Z-measurement.

- Other different bases are:

  ➢ $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, corresponding to the eigenstates of $\sigma_x$,

  ➢ $|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$, $|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$, corresponding to the eigenstates of $\sigma_y$.

# *Measurement*

- **Born rule**: the probability that a state $|\psi\rangle$ collapses during a project measurement onto the basis $\left\{|X\rangle, \ \left|X^\perp\right\rangle\right\}$ to the state $|X\rangle$ is given by $P(X) = |\langle X|\psi\rangle|^2$, $\sum_i P(X_i) = 1$

- Examples:
  - $|\psi\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \sqrt{2}|1\rangle)$ is measured in the basis $\{|0\rangle, |1\rangle\}$:
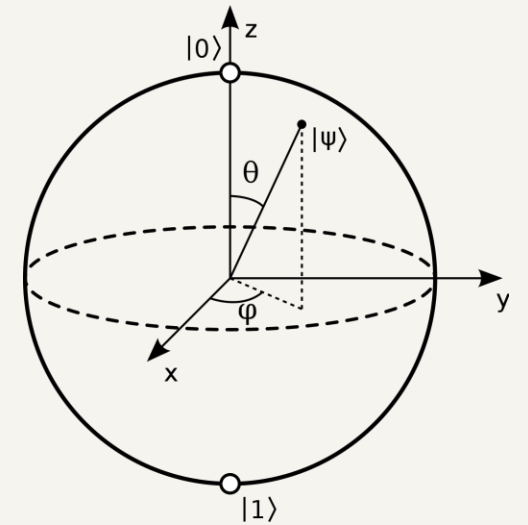
    $$P(0) = \left\langle 0 \left| \frac{1}{\sqrt{3}}(|0\rangle + \sqrt{2}|1\rangle)\right\rangle^2 = \left|\frac{1}{\sqrt{3}}\langle 0|0\rangle + \frac{\sqrt{2}}{\sqrt{3}}\langle 0|1\rangle\right|^2 = \frac{1}{3} \ \rightarrow \ P(1) = \frac{2}{3}$$

  - $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ is measured in the basis $\{|+\rangle, |-\rangle\}$:

    $$P(+) = |\langle +|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right|^2 = \frac{1}{4}|(\langle 0|0\rangle - \langle 0|1\rangle + \langle 1|0\rangle - \langle 1|1\rangle)|^2 = 0 \rightarrow \text{expected as } \langle +|-\rangle = 0,$$

    $$P(-) = |\langle -|-\rangle|^2 = 1$$

# *Bloch sphere*



- We can write any normalized **pure state** as $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$, where $\varphi \in [0, 2\pi]$ describes the relative phase and $\theta \in [0, \pi]$ determines the probability to measure $|0\rangle, |1\rangle$: $P(|0\rangle) = \cos^2\frac{\theta}{2}, P(|1\rangle) = \sin^2\frac{\theta}{2}$.

- All normalized pure states can be illustrated on the surface of a sphere with radius $|\vec{r}| = 1$, which we call the **Bloch sphere**.

- The coordinates of such a state are given by the Bloch vector: $\vec{r} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$

# *Bloch sphere*



examples:
- $|0\rangle$ : $\theta = 0$, $\varphi$ arbitrary $\rightarrow \vec{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
- $|1\rangle$ : $\theta = \pi$, $\varphi$ arb. $\rightarrow \vec{r} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$
- $|+\rangle$ : $\theta = \frac{\pi}{2}$, $\varphi = 0$ $\rightarrow \vec{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
- $|-\rangle$ : $\theta = \frac{\pi}{2}$, $\varphi = \pi$ $\rightarrow \vec{r} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$
- $|+i\rangle$ : $\theta = \frac{\pi}{2}$, $\varphi = \frac{\pi}{2}$ $\rightarrow \vec{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
- $|-i\rangle$ : $\theta = \frac{\pi}{2}$, $\varphi = \frac{3\pi}{2}$ $\rightarrow \vec{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
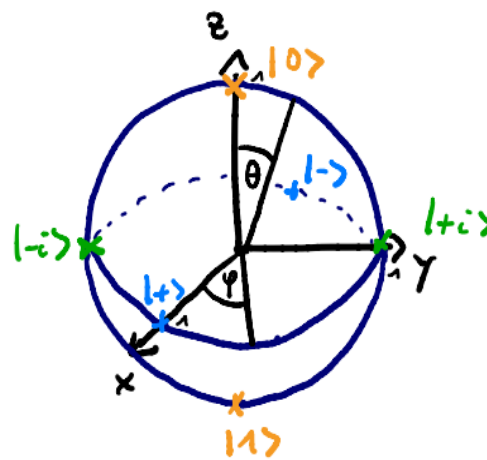
- **Be careful**: On the Bloch sphere, angles are twice as big as in Hilbert space:
  - ➢ e.g., $|0\rangle$ & $|1\rangle$ are orthogonal, but on the Bloch sphere their angle is 180º.
  - ➢ For a general state, $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \cdots \rightarrow \theta$ is the angle on the Bloch sphere, while $\frac{\theta}{2}$ is the actual angle in Hilbert space!

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Photo courtesy of : IBM quantum summer school 2019

# *Quantum circuits: single qubit gates*

- **Circuit model**: sequence of building block that carry out computations, called **gates**.

input ——— algorithm ——— output

- **Quantum gates** are represented by unitary matrices, A unitary matrix is a square matrix of complex numbers, whose inverse is equal to its conjugate transpose.

- **Single qubit gates**:

Hadamard $-\boxed{H}-$ $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ ← creates superposition

rotation around X-axis by $\pi$ → Pauli-$X$ $-\boxed{X}-$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ← bit flip

rotation around Y-axis by $\pi$ → Pauli-$Y$ $-\boxed{Y}-$ $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ ← bit & phase flip

rotation around Z-axis by $\pi$ → Pauli-$Z$ $-\boxed{Z}-$ $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ ← phase flip

Phase $-\boxed{S}-$ $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ ← used to change from Z to Y-basis

$\pi/8$ $-\boxed{T}-$ $\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$

# Quantum circuits: single qubit gates

- $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|$

  $\hookrightarrow \sigma_x |0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$ , $\sigma_x |1\rangle = (|0\rangle\langle 1| + |1\rangle\langle 0|) \cdot |1\rangle) = |0\rangle\underbrace{\langle 1|1\rangle}_{1} + |1\rangle\underbrace{\langle 0|1\rangle}_{0} = |0\rangle$

- $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$

  $\hookrightarrow \sigma_z |+\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = |-\rangle$ ,

  $\sigma_z |-\rangle = (|0\rangle\langle 0| - |1\rangle\langle 1|) \cdot \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$
  $= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$

- Hadamard gate : one of the most important gates for quantum circuits

  $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$

  $\hookrightarrow H|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = |+\rangle$ , $H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|) \cdot |1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$

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# *Quantum circuits: multipartite quantum states*
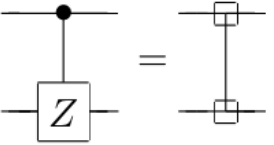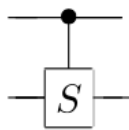
- We use tensor product to describe multiple states:

  ➢ $|a\rangle \otimes |b\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix}$

  ➢ Example: system A is in state $|1\rangle_A$ and system B is in state $|0\rangle_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, states of this form are called **uncorrelated**.

  ➢ But there are also bipartite states that cannot be written as $|\psi\rangle_a \otimes |\psi\rangle_b$. These states are **correlated** and sometimes even entangled (very strong correlation), e.g. $|\psi\rangle_{AB}^{(00)} = \frac{1}{\sqrt{2}} (|00\rangle_{AB} + |11\rangle_{AB}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, it so called Bell state, used for teleportation, cryptography, Bell tests, etc.

# Quantum circuits: multiple-qubit gates

# *Quantum circuits: two-qubit gates*

- Classical example: XOR

X
Y
XOR — X⊕Y      **irreversible**: given the output, we cannot recover the input.

- But as quantum theory is unitary, we only consider unitary and therefore reversible gates
- Quantum example: CNOT gate

$a \longrightarrow a' = a$

$b \longrightarrow b' = a \oplus b$

| a | b | a' | b' |
|---|---|----|----|
| 0 | 0 | 0  | 0  |
| 0 | 1 | 0  | 1  |
| 1 | 0 | 1  | 1  |
| 1 | 1 | 1  | 0  |
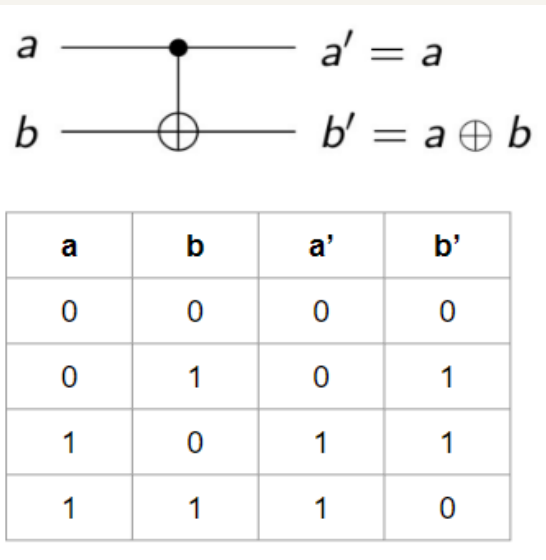
Quantum circuits can perform all function that can be calculated classically.

# *Entanglement*

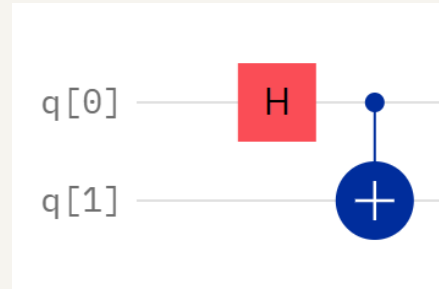- If a pure state $|\psi\rangle_{AB}$ on system A,B cannot be written as $|\psi\rangle_a \otimes |\phi\rangle_b$, it is entangled.
- These are four so called **Bell states** that are maximally entangled and build on orthonormal basis:
  - $\left|\psi^{00}\right\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$,
  - $\left|\psi^{01}\right\rangle := \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$,
  - $\left|\psi^{10}\right\rangle := \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$,
  - $\left|\psi^{11}\right\rangle := \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

# *Entanglement*

- Creation of Bell states:



$$|q_0q_1\rangle_{00} \; H_0 \rightarrow \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \; CNOT_{01} \rightarrow \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\psi^{00}\rangle,$$

$$|q_0q_1\rangle_{01} \; H_0 \rightarrow \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle) \; CNOT_{01} \rightarrow \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = |\psi^{01}\rangle,$$

$$|q_0q_1\rangle_{10} \; H_0 \rightarrow \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle) \; CNOT_{01} \rightarrow \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = |\psi^{10}\rangle,$$

$$|q_0q_1\rangle_{11} \; H_0 \rightarrow \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle) \; CNOT_{01} \rightarrow \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = |\psi^{11}\rangle.$$
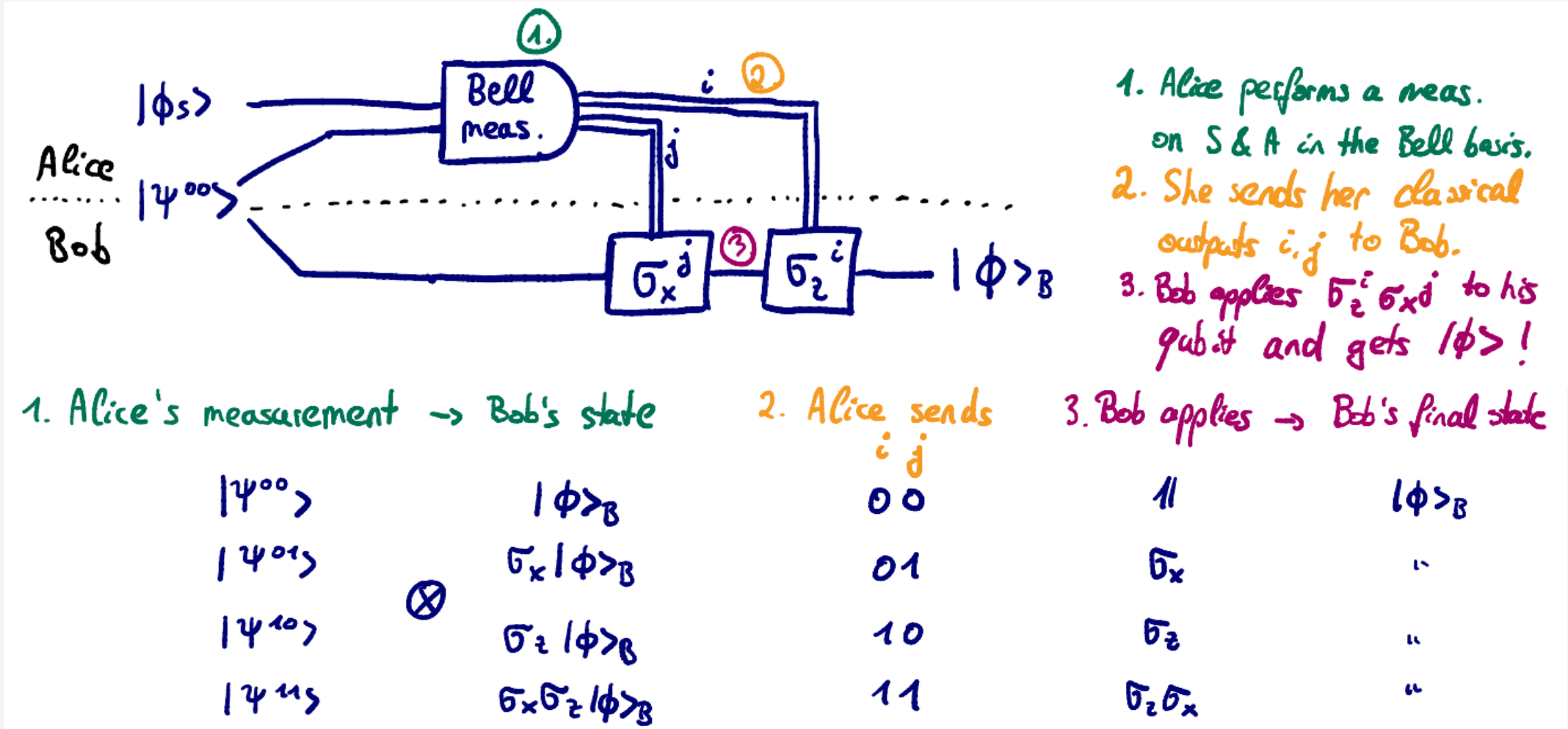
# *Teleportation*

- Goal:
  - ➤ Alice want to send her (unknown) state $|\phi\rangle_s := \alpha|0\rangle_s + \beta|1\rangle_s$ to Bob.
  - ➤ She can only send him two classical bits though.
  - ➤ They both share the maximally entangled state $|\psi\rangle_{AB}^{(00)} = \frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB})$.

- Initial states of the total system:

$$|\phi\rangle_s \otimes |\psi^{00}\rangle_{AB} = \frac{1}{\sqrt{2}}\left(\alpha|000\rangle_{SAB} + \alpha|011\rangle_{SAB} + \beta|100\rangle_{SAB} + \beta|111\rangle_{SAB}\right)$$

$$= \frac{1}{2\sqrt{2}}\Big[ \left(|00\rangle_{SA} + |11\rangle_{SA}\right) \otimes \left(\alpha|0\rangle_B + \beta|1\rangle_B\right) + \left(|01\rangle_{SA} + |10\rangle_{SA}\right) \otimes \left(\alpha|1\rangle_B + \beta|0\rangle_B\right)$$

$$+ \left(|00\rangle_{SA} - |11\rangle_{SA}\right) \otimes \left(\alpha|0\rangle_B - \beta|1\rangle_B\right) + \left(|01\rangle_{SA} - |10\rangle_{SA}\right) \otimes \left(\alpha|1\rangle_B - \beta|0\rangle_B\right)\Big]$$

$$= \frac{1}{2}\Big[ |\psi^{00}\rangle_{SA} \otimes |\phi\rangle_B + |\psi^{01}\rangle_{SA} \otimes \left(\sigma_x|\phi\rangle_B\right)$$

$$+ |\psi^{10}\rangle_{SA} \otimes \left(\sigma_z|\phi\rangle_B\right) + |\psi^{11}\rangle_{SA} \otimes \left(\sigma_x\sigma_z|\phi\rangle_B\right)\Big]$$

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# Teleportation

- Protocol:

① Bell meas.

Alice

Bob

$|\phi_s\rangle$

$|\psi^{00}\rangle$

② $i$ $j$

③ $\sigma_x^j$ $\sigma_z^i$ — $|\phi\rangle_B$

1. Alice performs a meas. on S & A in the Bell basis.
2. She sends her classical outputs $i, j$ to Bob.
3. Bob applies $\sigma_z^i \sigma_x^j$ to his qubit and gets $|\phi\rangle$ !

1. Alice's measurement → Bob's state

| | | |
|---|---|---|
| $|\psi^{00}\rangle$ | | $|\phi\rangle_B$ |
| $|\psi^{01}\rangle$ | $\otimes$ | $\sigma_x|\phi\rangle_B$ |
| $|\psi^{10}\rangle$ | | $\sigma_z|\phi\rangle_B$ |
| $|\psi^{11}\rangle$ | | $\sigma_x\sigma_z|\phi\rangle_B$ |

2. Alice sends $i$ $j$  |  3. Bob applies → Bob's final state

| $i\,j$ | | |
|---|---|---|
| 0 0 | 1 | $|\phi\rangle_B$ |
| 0 1 | $\sigma_x$ | " |
| 1 0 | $\sigma_z$ | " |
| 1 1 | $\sigma_z\sigma_x$ | " |

- Alice's state collapsed during the measurement, so she doesn't have the initial state $|\phi\rangle_s$ anymore. This is expected due to the no-cloning theorem, as she cannot copy her state, but just send her state to Bob when destroying her own.
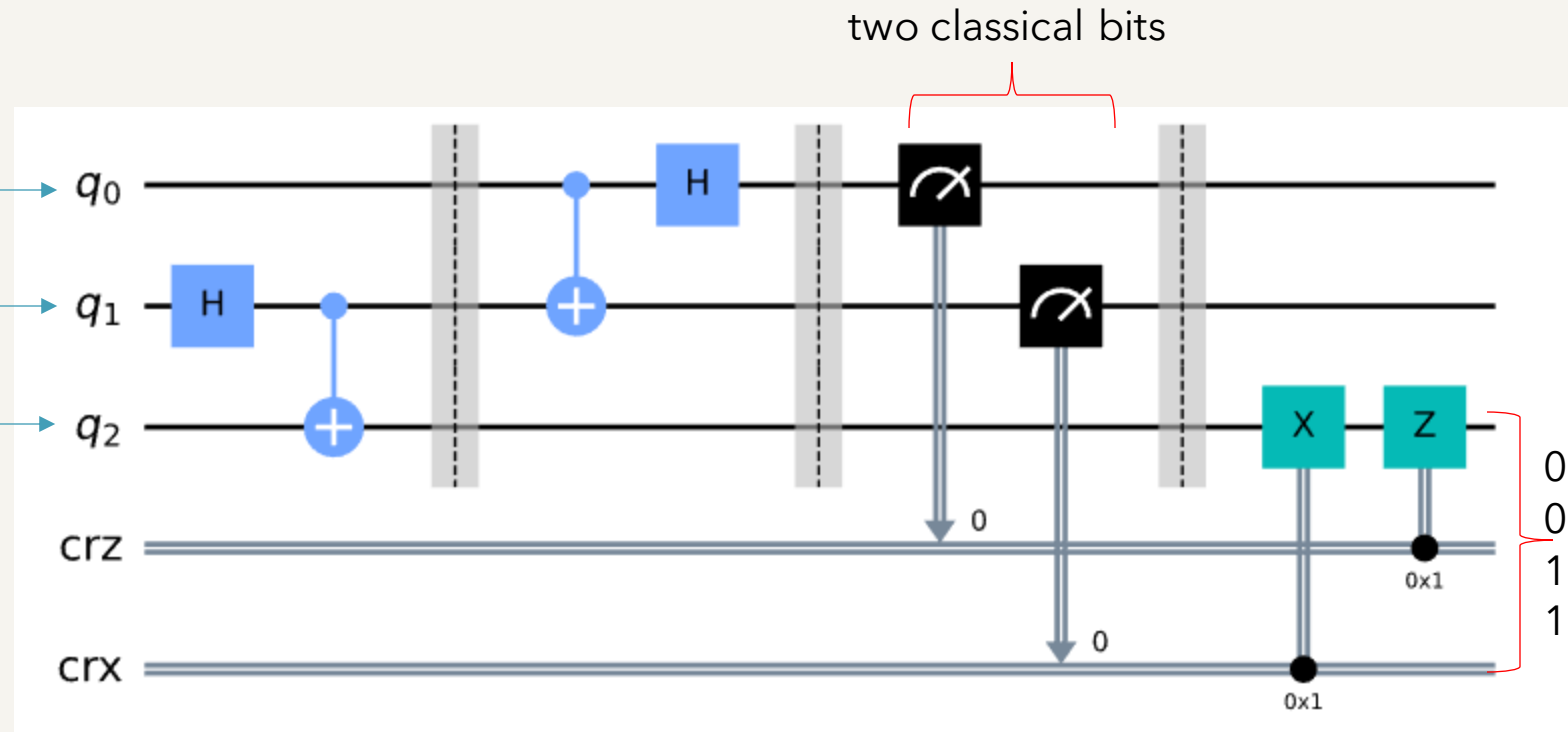
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Photo courtesy of : IBM quantum summer school 2019

# *Teleportation*

- Quantum circuit:



The qubit she is trying to send Bob.

Alice's qubit

Bob's qubit

two classical bits

00 → Do nothing
01 → Apply X gate
10 → Apply Z gate
11 → Apply ZX gate

# *Superdense coding*

- Superdense coding is a procedure that allows someone to send two classical bits to another party using just a single qubit of communication.

- Take advantage of quantum mechanics to more efficiently transmit classical information.

- Word "coding" means there are 2 essential processes, encoding and decoding:

  ➢ encoding: classical state → quantum state,

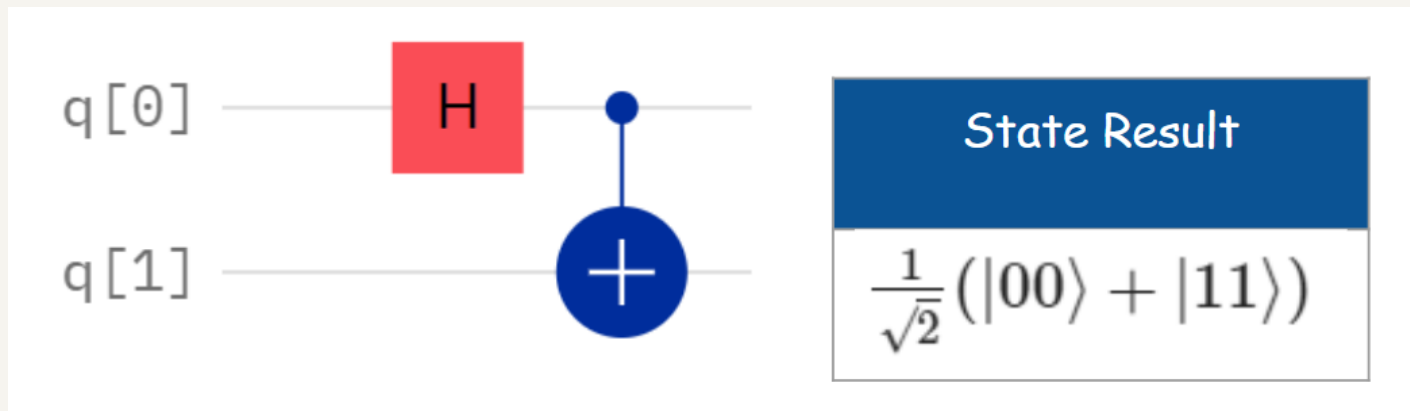  ➢ decoding: quantum state → classical state.

| Teleportation | Superdense Coding |
|---|---|
| Transmit **one qubit** using **two classical bits**. | Transmit **two classical bits** using **one qubit**. |

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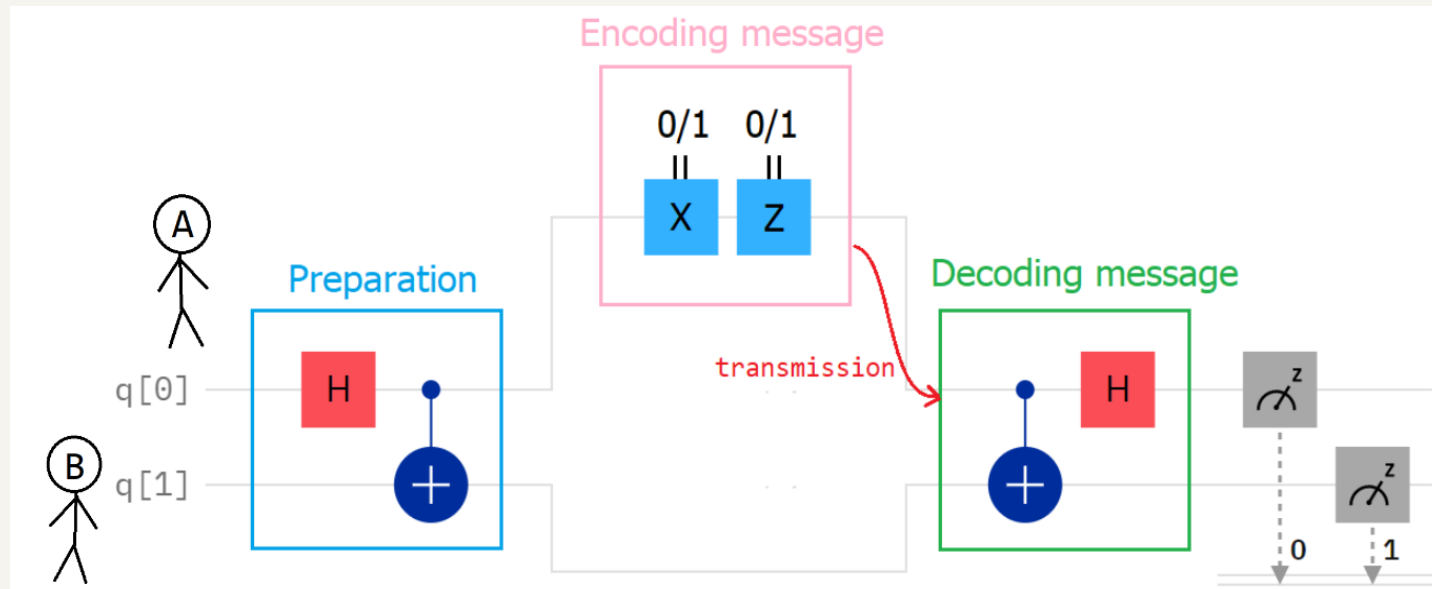
# *Superdense coding*

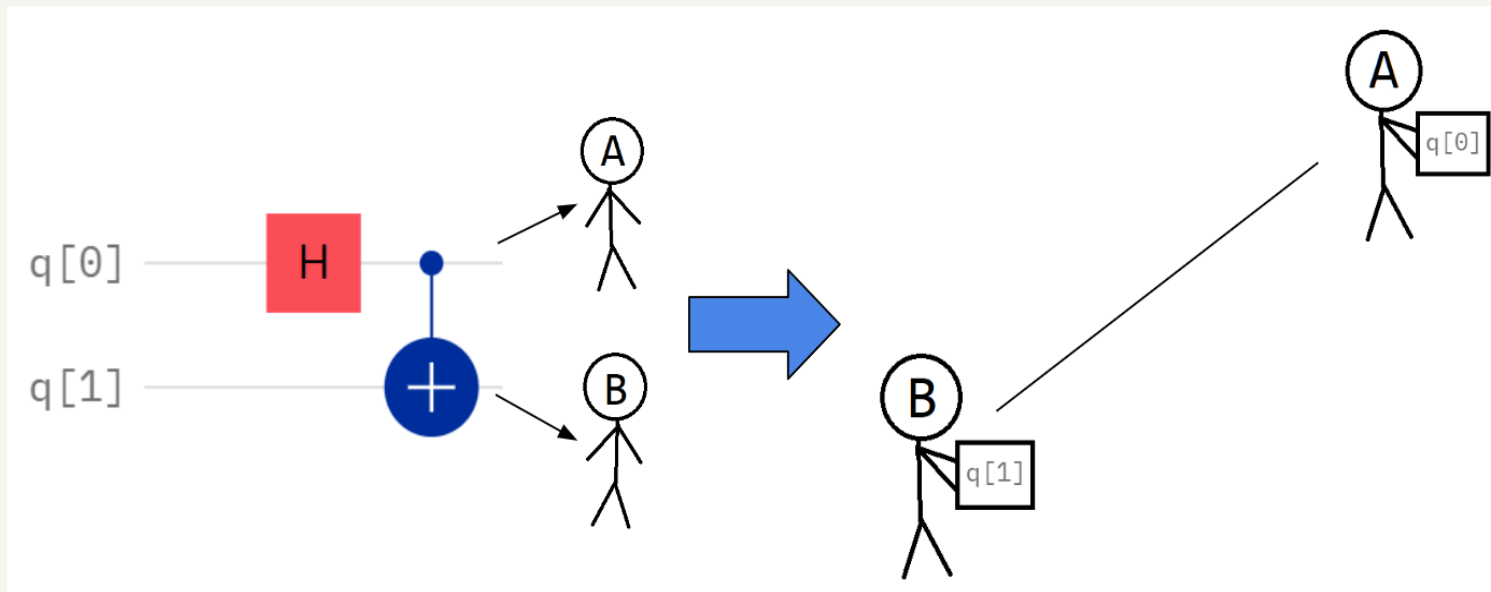

- Superdense coding includes 4 steps:
  - ➤ preparation,
  - ➤ encoding message,
  - ➤ transmission,
  - ➤ decoding message.

# *Superdense coding*



$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

- Step 1: preparation
  - ➤ Start with 2 qubits in the basis state |0⟩.
  - ➤ Applying Hadamard gate to the first qubit and CNOT gate (the first qubit as control, another qubit as target) accordingly.
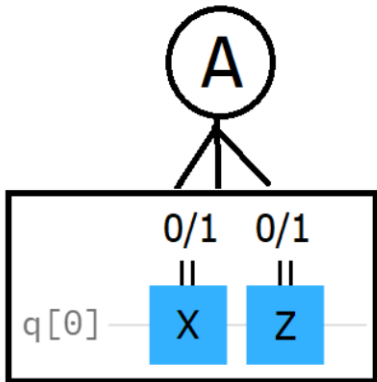
# *Superdense coding*



- Step 1: preparation
  - ➤ Give the first qubit to A and the second qubit to B.
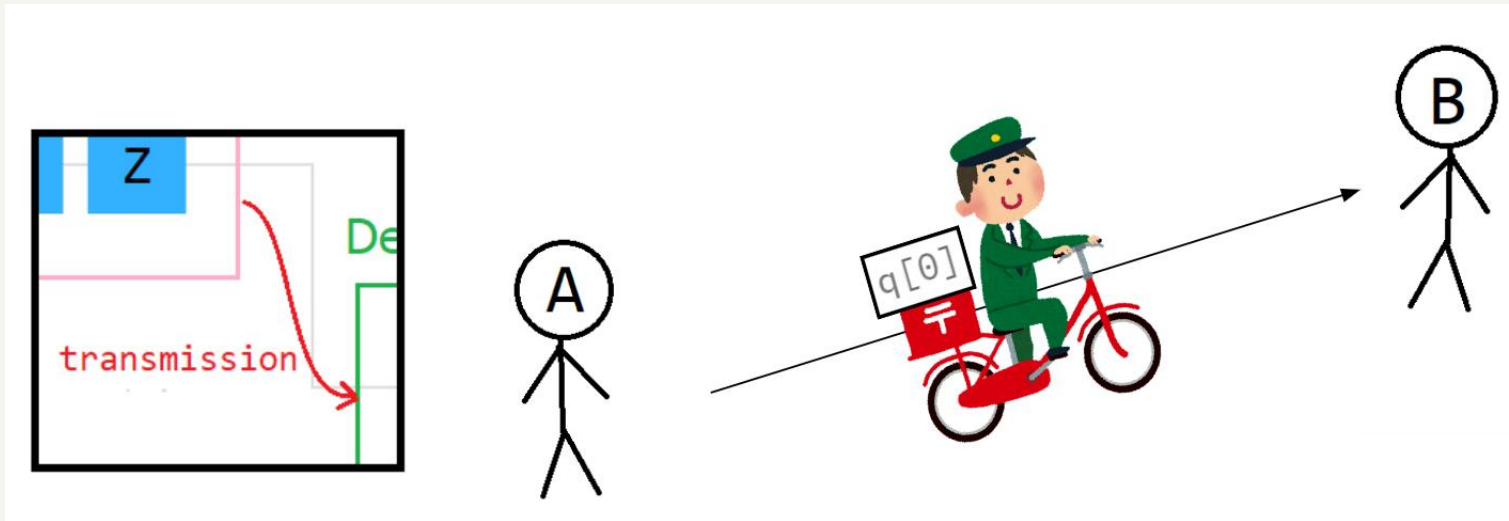  - ➤ A and B travel far away.

# Superdense coding



| Message | Applied Gate | State Result |
|---------|--------------|--------------|
| 00 | I | $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ |
| 01 | X | $\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$ |
| 10 | Z | $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$ |
| 11 | ZX | $\frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$ |

| Message | Applied Gate |
|---------|--------------|
| 00 | |
| 01 | X |
| 10 | Z |
| 11 | X Z |

- Step 2: encoding message
  - ➢ A encodes the classical state in the qubit by applying gate(s).

# *Superdense coding*



- Step 3: transmission
  - ➤ A sends the first qubit to B.

# *Superdense coding*
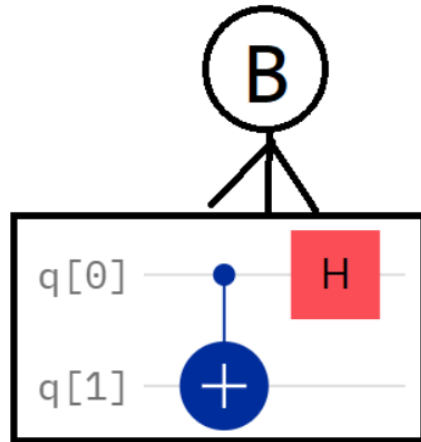


| Message | State Result |
|---------|--------------|
| 00 | $|00\rangle$ |
| 01 | $|01\rangle$ |
| 10 | $|10\rangle$ |
| 11 | $|11\rangle$ |

- Step 4: decoding message
  - ➢ Applying CNOT gate (the first qubit as control, another qubit as target) and Hadamard gate to the first qubit accordingly.
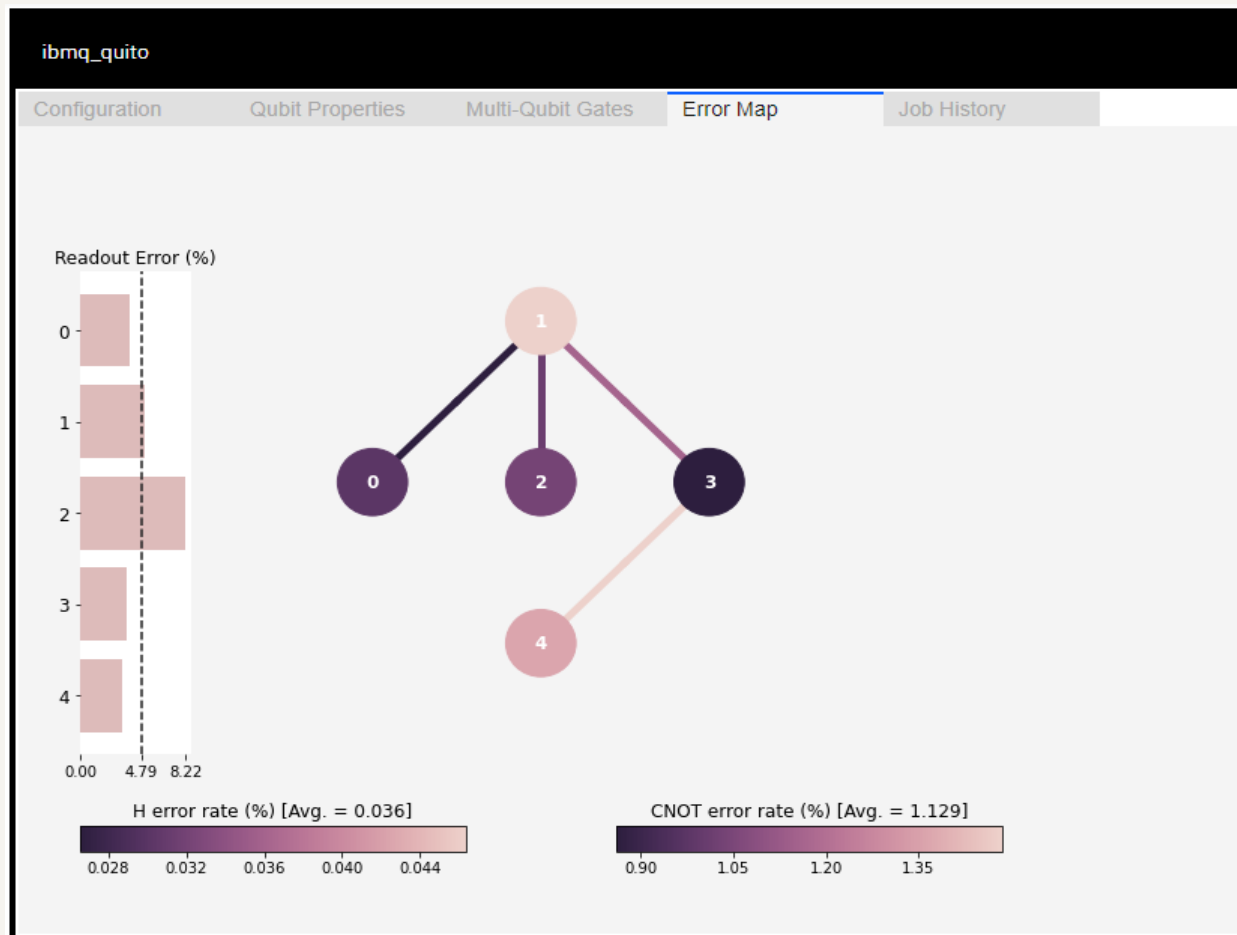
# *Superdense coding*

Test the circuit which encodes message "**11**" and run on "**ibm_oslo**".
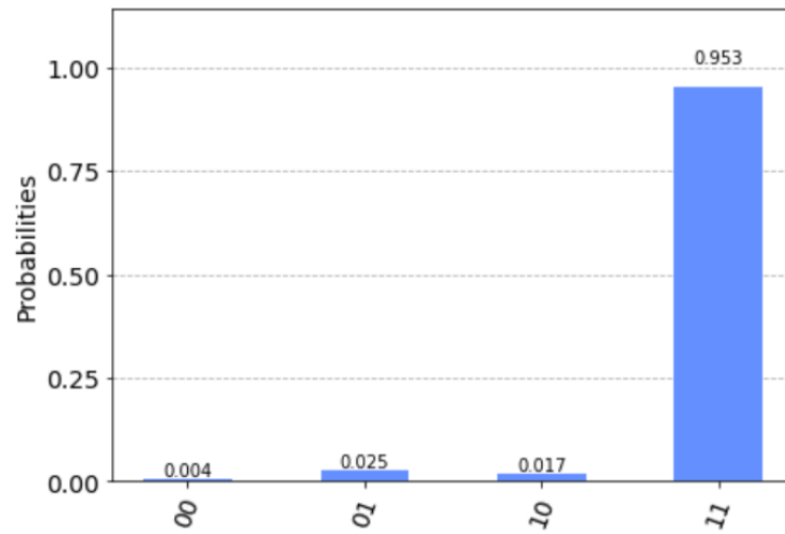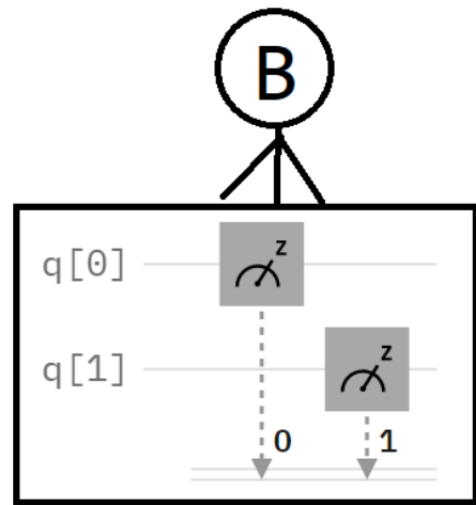


- Step 4: decoding message
  - ➤ Finally, measure both qubits.

# *How the noise properties affect the result*



- There are often optimizations that the transpiler can perform that reduce the overall gate count, and thus total length of the input circuits.

- Qiskit library has a command "backend" to show the chosen backend information graphically such as "Error Map".

- We can select a good initial layout considering connectivity and error information that you can find from the map to initial layout onto the physical qubits with at least noise.