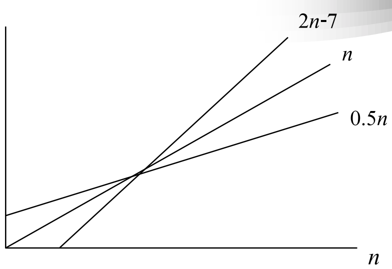
Analysis of Algorithms

Growth of Functions

เนื้อหา

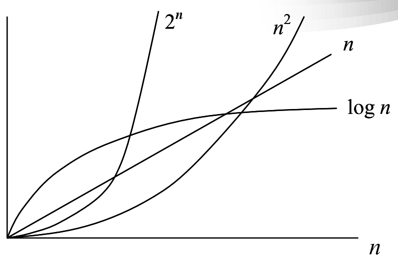
- Growth of Functions
- Asymptotic Notation : $O, \Omega, \Theta, o, \omega$
- Asymptotic Notation Properties

Growth of Functions



Linear

Growth Rates



Growth Rates

$$\lim_{n \to \infty} f(n) / g(n) = \begin{cases} 0 & f(n) \text{ grows slower than } g(n) \\ \infty & f(n) \text{ grows faster than } g(n) \\ \text{otherwise} & f(n) \text{ and } g(n) \text{ have the same growth rate} \end{cases}$$

$f(n) \prec g(n)$

- $\lim_{n \to \infty} f(n) / g(n) = 0$

$f(n) \prec g(n)$: $f(n)$ grows slower than $g(n)$

$$0.5^n \prec 1 \prec \log n \prec \log^6 n \prec n^{0.5} \prec n^3 \prec 2^n \prec n!$$

l'Hôpital's Rule

If $f(n)$ and $g(n)$ are differentiable, $\lim_{n \to \infty} f(n) = \infty$,

$\lim_{n \to \infty} g(n) = \infty$, and $\lim_{n \to \infty} f'(n)/g'(n)$ exists, then

$$\lim_{n \to \infty} f(n)/g(n) = \lim_{n \to \infty} f'(n)/g'(n)$$

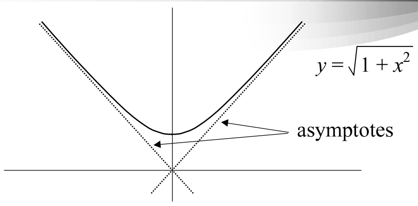
lg n vs. $\sqrt{n}$

$f(n) = \lg n \quad g(n) = \sqrt{n}$

$$\begin{aligned}\lim_{n\to\infty} \frac{\lg n}{\sqrt{n}} &= \lim_{n\to\infty} \frac{\ln n}{\ln 2 \sqrt{n}} \\ &= \frac{1}{\ln 2} \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} \\ &= \frac{1}{\ln 2} \lim_{n\to\infty} \frac{1/n}{1/\left(2\sqrt{n}\right)} \\ &= \frac{1}{\ln 2} \lim_{n\to\infty} \frac{2}{\sqrt{n}} \\ &= 0\end{aligned}$$

Every sublinear function grows faster than any polylogarithmic function e.g., $n^{0.01}$ vs. $(\log n)^{100}$

Asymptotic

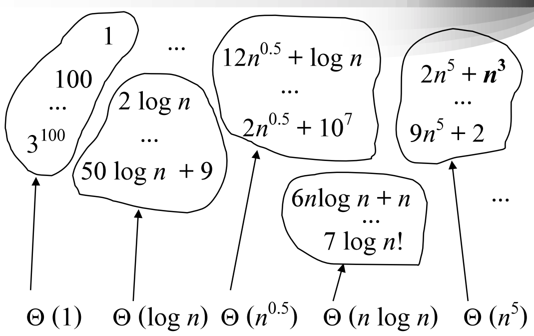


Asymptotic : any approximation value that gets closer and closer to the truth, when some parameter approaches a limiting value.

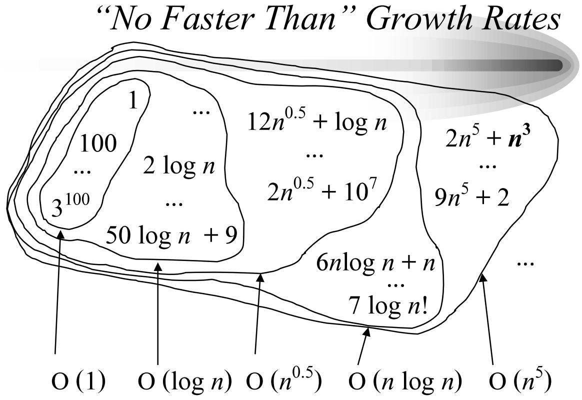
Asymptotic Notations

- Deal with the behaviour of functions in the limit (for sufficiently large value of its parameters)
- Permit substantial simplification (napkin mathematic, rough order of magnitude)
- Classify functions by their growth rates

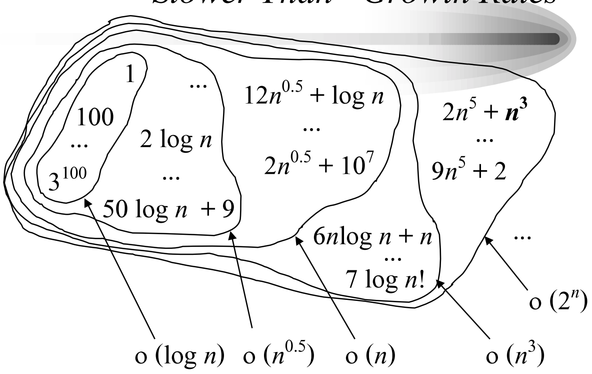
"Same" Growth Rates



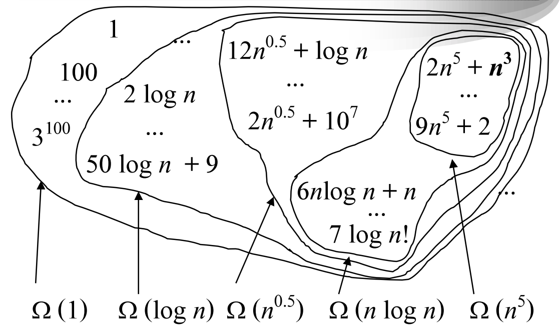
"No Faster Than" Growth Rates



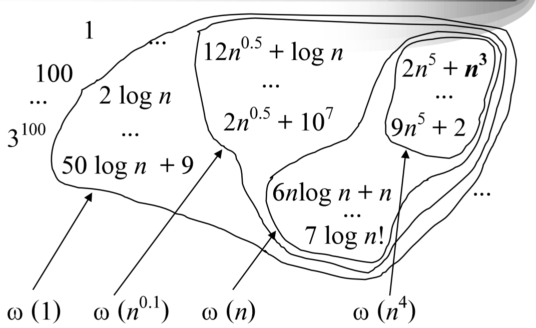
"Slower Than" Growth Rates



"No Slower Than" Growth Rates



"Faster Than" Growth Rates



Special Orders of Growth

- constant : $\Theta(1)$
- logarithmic : $\Theta(\log n)$
- polylogarithmic : $\Theta(\log^c n)$, $c \ge 1$
- sublinear : $\Theta(n^a)$, $0 < a < 1$
- linear : $\Theta(n)$
- quadratic : $\Theta(n^2)$
- polynomial : $\Theta(n^c)$, $c \ge 1$
- exponential : $\Theta(c^n)$, $c > 1$

Analogy

- $f(n)$ and $g(n)$ — f and g (real numbers)
- $f(n) = \Theta(g(n))$ $\approx$ $f = \boldsymbol{g}$
- $f(n) = O(g(n))$ $\approx$ $f \le \boldsymbol{g}$
- $f(n) = o(g(n))$ $\approx$ $f < \boldsymbol{g}$
- $f(n) = \Omega(g(n))$ $\approx$ $f \ge \boldsymbol{g}$
- $f(n) = \omega(g(n))$ $\approx$ $f > \boldsymbol{g}$

Not all functions are asymptotically comparable

n vs. $n^{1+\sin n}$

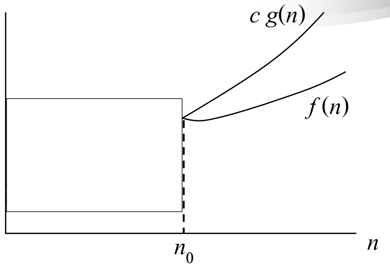
O-notation

$O(g(n)) = \{ f(n)$: there exist positive constants c and n_0 such that $0 \le f(n) \le c\, g(n)$ for all $n \ge n_0 \}$

$27n^2 + 0.5n \in O(n^3)$?

$27n^2 + 0.5n \le ???\, n^3$ for all $n \ge ?$

Asymptotic Upper Bound



$f(n) = \mathrm{O}(\,g(n)\,)$

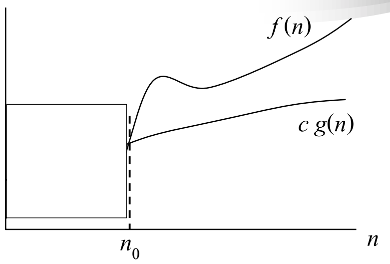
Ω-notation

$\Omega(\,g(n)) = \{\,f(n) :$ there exist positive constants c and n_0 such that $0 \le c\,g(n) \le f(n)$ for all $n \ge n_0$ $\}$

$27n^2 + 0.5n \in \Omega(\,n\,)$?

$???n \le 27n^2 + 0.5n$ for all $n \ge ?$

Asymptotic Lower Bound



$f(n) = \Omega(\,g(n)\,)$

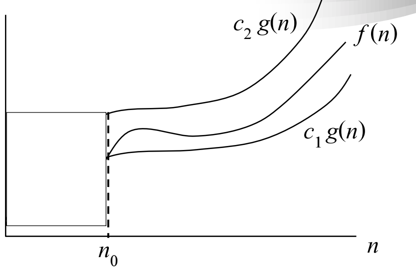
Θ-notation

$\Theta(\,g(n)) = \{\,f(n) :$ there exist positive constants c_1, c_2, and n_0 such that $0 \le c_1 g(n) \le f(n) \le c_2\,g(n)$ for all $n \ge n_0$ $\}$

$27n^2 + 0.5n \in \Theta(\,n^2\,)$?

$???\,n^2 \le 27n^2 + 0.5n \le ???\;n^2$ for all $n \ge ?$

Asymptotic Tight Bound



$f(n) = \Theta(\,g(n)\,)$

o-notation and ω-notation

$f(n) = \mathrm{o}(\,g(n)\,)$

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

$f(n) = \omega(\,g(n)\,)$

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$$

Asymptotic Notation Properties

- Transitivity
- Reflexivity
- Symmetry
- Transpose symmetry

Transitivity

- $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$ imply $f(n) = \Theta(h(n))$
- $f(n) = \mathrm{O}(g(n))$ and $g(n) = \mathrm{O}(h(n))$ imply $f(n) = \mathrm{O}(h(n))$
- $f(n) = \Omega(g(n))$ and $g(n) = \Omega(h(n))$ imply $f(n) = \Omega(h(n))$
- $f(n) = o(g(n))$ and $g(n) = o(h(n))$ imply $f(n) = o(h(n))$
- $f(n) = \omega(g(n))$ and $g(n) = \omega(h(n))$ imply $f(n) = \omega(h(n))$

Reflexivity

- $f(n) = \Theta(\, f(n)\,)$
- $f(n) = \mathrm{O}(\, f(n)\,)$
- $f(n) = \Omega(\, f(n)\,)$

Symmetry and Transpose Symmetry

- $f(n) = \Theta(\, g(n)\,)$ if and only if $g(n) = \Theta(\, f(n)\,)$
- $f(n) = \mathrm{O}(\, g(n)\,)$ if and only if $g(n) = \Omega(\, f(n)\,)$
- $f(n) = o(\, g(n)\,)$ if and only if $g(n) = \omega(\, f(n)\,)$

Θ, Ω, O

- $f(n) = \Theta(\, g(n)\,)$ if and only if
  - $f(n) = \Omega(\, g(n)\,)$ and
  - $f(n) = \mathrm{O}(\, g(n)\,)$

Example

$$\sum_{i=1}^{n} i^k = \Theta\left(n^{k+1}\right)$$

O

$$\sum_{i=1}^{n} i^k \le \sum_{i=1}^{n} n^k = n^{k+1} = \mathrm{O}\left(n^{k+1}\right)$$

Ω

$$\sum_{i=1}^{n} i^k \ge \sum_{i=\lceil n/2 \rceil}^{n} i^k \ge \sum_{i=\lceil n/2 \rceil}^{n} \left(\frac{n}{2}\right)^k \ge \frac{n}{2}\left(\frac{n}{2}\right)^k = \frac{n^{k+1}}{2^{k+1}} = \Omega\left(n^{k+1}\right)$$

Example

$\log n! = \Theta(n \log n)$

O

$$
\begin{aligned}
n! &= n \times (n-1) \times \ldots \times 2 \times 1 \\
&\le n \times n \times \ldots \times n \times n = n^n \\
\log n! &\le \log n^n \\
&= O(n \log n)
\end{aligned}
$$

Ω

$$
\begin{aligned}
n! &= n \times (n-1) \times \ldots \times (n/2) \times (n/2-1) \times \ldots \times 2 \times 1 \\
&\ge n/2 \times n/2 \times \ldots \times n/2 \times 1 \times \ldots \times 1 \times 1 \\
&\ge (n/2)^{n/2} \\
\log n! &\ge (n/2) \log (n/2) \\
&= \Omega(n \log n)
\end{aligned}
$$

Logarithms

- Asymptotically, in logarithm,
  - the base of the log does not matter

$\log_b n = (\log_c n) / (\log_c b)$

$\log_{10} n = (\log_2 n) / (\log_2 10) = \Theta(\log n)$

  - any polynomial function of n does not matter

$\log n^{30} = 30 \log n = \Theta(\log n)$

- lg, ln, log

(CS) (math) (asymptotic)

Polylog, Polynomial, Exponential

- Any positive exponential function grows faster than any polynomial function

$f(n)$: monotonically growing function,

$(f(n))^c = o(a^{f(n)})$, $c > 0$, $a > 1$

- Any positive polynomial function grows faster than any polylogarithmic function

$\log^b n = o(n^c)$

One-Way Equality

- $n = O(n^2) \implies n \in O(n^2)$
- $\cancel{O(n^2) = n}$
- Why don't we use ∈ ? (GKP)
  - tradition : the practice stuck
  - tradition : CS is used to abuse equal sign
  - tradition : read "=" as "is", "is" is one-way
  - natural : when we do asymptotic calculation

Asymptotic Notaton in Equations

- $H_n = 1/1 + 1/2 + \ldots + 1/\boldsymbol{n}$

$= \ln n + \gamma + O(1/n)$

- $2n^2 + 3n + 1 = 2n^2 + \Theta(n)$

$2n^2 + 3n + 1 = 2n^2 + f(n), \quad f(n) = \Theta(n)$

- Eliminate inessential detail

e.g., MergeSort : $T(n) = 2T(n/2) + n$?

$T(n) = 2T(n/2) + (n-1)$?

$T(n) = 2T(n/2) + \Theta(n) = \Theta(n \log n)$

Manipulating Asymptotic Notations

- $c\, O(f(n)) = O(f(n))$
- $O(O(f(n))) = O(f(n))$
- $O(f(n))\, O(g(n)) = O(f(n)\, g(n))$
- $O(f(n)\, g(n)) = f(n)\, O(g(n))$
- $O(f(n) + g(n)) = O(\max(f(n), g(n)))$

$$\sum_{k=1}^{n} O(f(k)) = O\left(\sum_{k=1}^{n} f(k)\right)$$

Example : BuildHeap

$$k = \lfloor \lg n \rfloor$$

$$\sum_{h=0}^{k} \left(\frac{n}{2^h} O(h) \right) = O\left(n \sum_{h=0}^{k} \frac{h}{2^h} \right)$$

$$= O\left(n \sum_{h=0}^{\infty} \frac{h}{2^h} \right)$$

$$= O(2n)$$

$$= O(n)$$

Conclusion

- Growth of functions
  - give a simple characterization of function's behavior
  - allow us to compare the relative growth rates of functions
- Use asymptotic notation to classify functions by their growth rates
- Asymptotics is the art of knowing where to be sloppy and where to be precise

http://www.cp.eng.chula.ac.th/faculty/spj