Counting

- Basic
- Pigeonhole Principle
- Recurrence Relation
- Generating Function
- Inclusion and Exclusion

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Books:

- Rosen, Discrete Mathematics and its applications
- Somchai Prasitjutragul, Discrete Mathematics
- A. Tucker, Applied Combinatorics 3rd edition

$\boxed{\text{What do we count?}}$

- Number of ways we can arrange wedding photos
- Number of ways to select 3 kinds of fruit out of a basket that contains 10 kinds of fruit
- ...etc.

$\boxed{\text{The Sum Rule}}$

If there are 2 independent tasks, the first task can be done n1 ways, the second task can be done n2 ways,

then the number of ways to do BOTH tasks = $n1 + n2$.

Example: choose 1 teacher from 2 rooms. The first room has 5 teachers, the second room has 7 teachers.

The total number of ways is therefore $7+5 = 12$.

The Product Rule

If a procedure can be divided into steps n1 and n2, then the number of ways to do the entire procedure $= n1 * n2$.

Example:

- Choose 2 teachers, one from room A (5 teachers), the other from room B (7 teachers), to form a committee of 2.

  The total number of ways =5*7 = 35.

- How many ways to answer 10 multiple choice questions, when each question has 4 holes to tick. Assume that only 0 or 1 hole may be ticked.

  The answer is 5^{10}, because we include the possibilities of not ticking questions.

- How many licence plate can be formed from 3 english letters followed by 3 digits?

  Answer is $26^3 * 10^3$.

- Show that the number of subset of S = 2^n, where n is the size of S.

  We can represent the existence of each subset element using bit string. Each element either exists or does not exist (0 or 1). Therefore all the possibilities for n elements is 2^n

Using Sum and Product rule together

Example: How many possible passwords are there, if a password is allowed to be:

- 6-8 characters long
- each character is an uppercase or a digit
- at least one digit must be in each password

Answer = length6 + length7+length8

$$= (\underbrace{36^6}_{all-ways} - \underbrace{26^6}_{no-digits} + 36^7 - 26^7 + 36^8 - 26^8)$$

Inclusion and Exclusion

We must remember to delete the repeated counts.

Example: How many possible bit strings of length 8 $\underbrace{start - with - 1}_{1*2^7}$ or $\underbrace{end - with - 00}_{2^6*1*1}$.

The answer $= 2^7+2^6-2^5$, where 2^5 is the number of bit strings starting with 1 and end with 00.

Tree and Counting

1 branch = 1 possible choice

Example: Count the number of bit strings of length 3 that end with 0.

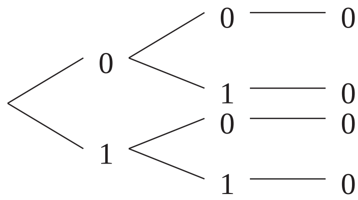

Figure 1: Counting, using tree

Counting Examples

1. How many bit strings are there of length 6 or less, (including the one with length 0)?

   Answer = longth0 + length1 + length2 + length3 + ...+ length6 $= 1 + 2 + 2^2 + \ldots + 2^6$

2. How many strings of 5 letters contain 'a' at least once?

   Answer $= 26^5 - \underbrace{25^5}_{no-'a'}$

3. How many integers from 100 to 999

   - are divisible by 7?

     Answer $= \lfloor 999/7 \rfloor - \lfloor 99/7 \rfloor = 128$

- have the same 3 decimal digits?

  Answer $= \underbrace{9}_{1st-can't-be-0} * 1 * 1$

- are not divisible by 4?

  Answer = All possibilities - divisible by 4

  $$= 9 * 10 * 10 - (\lfloor 999/4 \rfloor - \lfloor 99/4 \rfloor) = 675$$

- are divisible by 3 or 4?

  Answer = divisible by 3 + divisible by 4 - divisible by both 3 and 4 (i.e 12)

- are divisible by 3 but not 4?

  Answer = divisible by 3 - divisible by both

4. How many license plates (in english) can be made using 2 or 3 letters followed by 2 or 3 digits?

   Answer = 2 letters and 2 digits + 2 letters and 3 digits + 3 letters and 2 digits + 3 letters and 3 digits $= 26^2 * 10^2 + 26^2 * 10^3 + \ldots$

5. How many functions from set 1,2,...,n to 0,1:

   - are 1 to 1?

     Answer

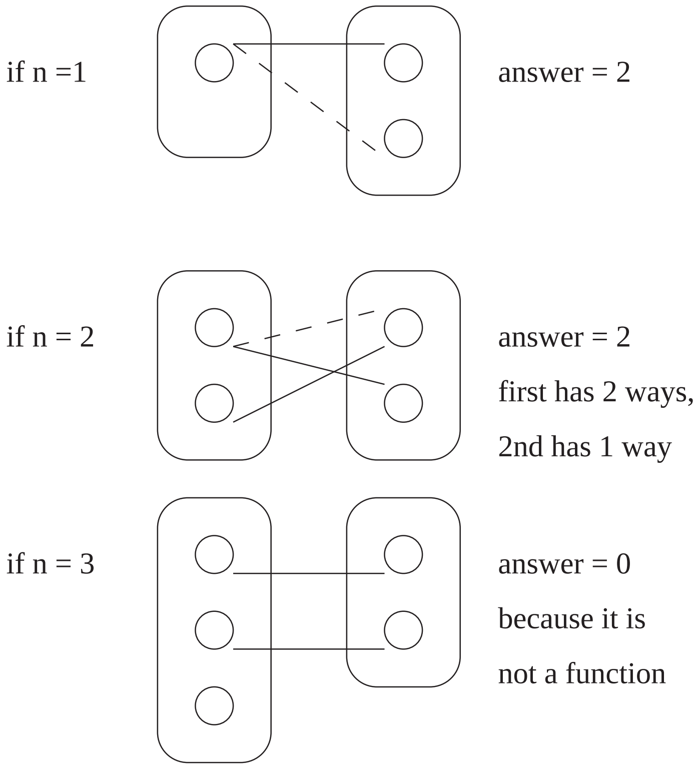

Figure 2: Various cases of Function 1-1

- assign 0 in the second set to n and 1 in the first set? Answer

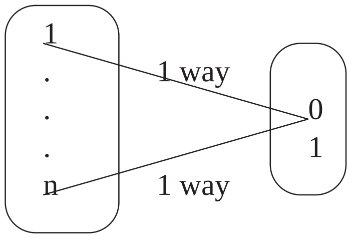

Figure 3: 0 is assigned to both n and 1

  - From the remaining n-2 elements, each will have two ways. Therefore the overall answer is $1 * 1 * 2^{n-2}$ if $n \geq 1$.
  - if $n = 1$, there is obviously 1 way in this case.

- are there so that, for positive integers in the first set that is less than n, 1 in the second set is assigned exactly to one of them? Answer

  - If $n = 1$, there will be no positive integer less than n, 1 can't be assigned to it. Therefore the answer is 0.
  - If $n = 2$, only 2 ways. The first element will have to be paired with 1. The second

element has two choices. So the overall answer is $2 * 1 = 2$ ways. (see figure 4)

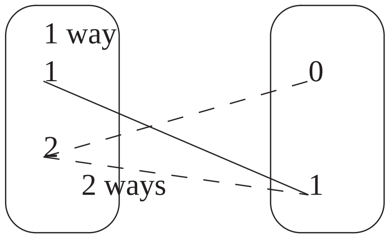

Figure 4: 1 is assigned to positive integer less than n, n=2

– If $n = n$, see figure 5.

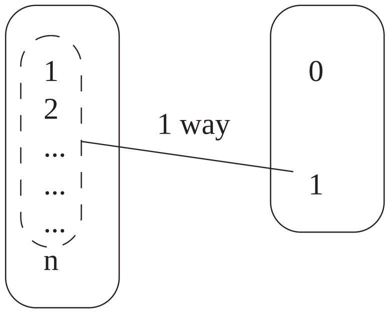

Figure 5: 1 is assigned to positive integer less than n, n=n

* The 1 in the second set has $n - 1$ choices to go.
* The rest of the first set (apart from n) have only 1 way each, to pair with 0.
* n itself has 2 ways, choosing between 0 and 1.
* Therefore the overall result is $(n-1) * 1 * 1 * \ldots * 1 * 2 = 2(n-1)$.

Pigeonhole Principle

<u>Theorems</u>

If k+1 or more objects are put in k boxes, then at least one box contains two or more objects.

If n objects are put in k boxes, then at least one box contains $\lceil n/k \rceil$ objects.

Example:

- 42 birds share 41 branches, there will be at least 1 branch with 2 birds.
- From 100 people, there are at least $\lceil 100/12 \rceil =$ 9 that were born in the same month.
- If students are given grades A to E, and we want at least 6 students to have the same grade, then $\lceil k/5 \rceil = 6$, i.e. k = 26. There must be at least 26 students.

During 30 days, a baseball team plays at least 1 game a day, but no more than 45 games after day 30. Show that there is 14 matches difference between two days in those 30 days.

Let a_i be the number of games played from day 1 to (and including) day i_{th}. Therefore:

$$1 \leq a_i \leq 45$$

We applied 14 that was given in the question:

$$15 \leq a_i + 14 \leq 59$$

Now we have $a_1, a_2, \ldots, a_{30}$ and $a_1 + 14, a_2 + 14, \ldots, a_{30} + 14$.

There are 60 of them. The possible number range that can be put for each of this is only 1 to 59. Therefore even though we try to assign each item with a different value, there will be 2 items with the same value. These 2 can't be from the same sequence because it is already determined that the value of items at 2 different days must differ at least 1 (plays at least 1 game a day). So we have:

$$a_{i1} = a_{i2} + 14$$

which is exactly what we are required to prove.

More Pigeonhole Examples

1. Show that, in a randomly selected 5 numbers, there are 2 numbers with the same remainder when divided by 4.

   When any number is divided by 4, there are only 4 possibilities for the remainder, 0 to 3.

   Therefore, with 5 numbers, surely one of them must produce the same remainder as one of the other four.

2. 6 computers, each connecting to at least one of the other computers. Show that there are at least 2 of them that connect to the same number of computers.

   the number of connections from each PC is 1 or 2 or ... or 5, i.e. 5 possibilities.

   Even though the first 5 PCs are connected to different number of PCs, the last one will have to connect to one of those 5 possibilities.

3. 25 students in the network class are either from computer engineering-undergrad, electrical engineering-undergrad, or master of computer engineering.

   - Prove that there are at least 9 students from computer engineering-undergrad, or at least 9 students from electrical engineering-undergrad, or at least 9 students from master of computer engineering.

     We prove by contradiction. Assume 8 or less for each group - the total can only amount to 24, contradicting the fact.

   - Prove that there are at least 3 students from computer engineering-undergrad, or at least 19 students from electrical engineering-undergrad, or at least 5 students master of computer engineering.

     We prove by contradiction again. Assume there are 2 or less from computer engineering-undergrad, and 18 or less from electrical engineering-undergrad, and 4 or less from master of computer engineering. The maximum total will be 24. Again, contradict with the original fact.

Related theorem- subsequence length

Every sequence of $n^2 + 1$ distinct real numbers contains at least one subsequence of length n+1 that is either

- strictly increasing, or
- strictly decreasing

Example:
8,11,9,1,4,6,12,10,5,7
10 terms $(3^2 + 1)$. By the above formula, there is at least one increasing/decreasing subsequence of length $3 + 1 = 4$
(1,4,6,12) (1,4,6,7) (1,4,5,7) (11,9,6,5)
Prove:
Let $a_1, a_2, \ldots, a_{n^2+1}$ be a sequence of n^2+1 distinct real numbers.
Let i_k be the length of the longest increasing subsequence starting at a_k.
Let d_k be the length of the longest decreasing subsequence starting at a_k.
Therefore a_k is associated with (i_k, d_k).
We prove by contradiction. Assume that there are no increasing or decreasing subsequence of length $n + 1$.

Then i_k and d_k are always $\leq n$. This means the maximum number of pairs (i_k, d_k) is n^2.

There are n^2 (i_k, d_k) that can associate with a_k, but the number of a_k is $n^2 + 1$.

Therefore, there must be 2 a_ks with the same associated (i_k, d_k). Let these be a_{k1} and a_{k2}.

If $a_{k1} < a_{k2}$, impossible situation arises, see figure 6.

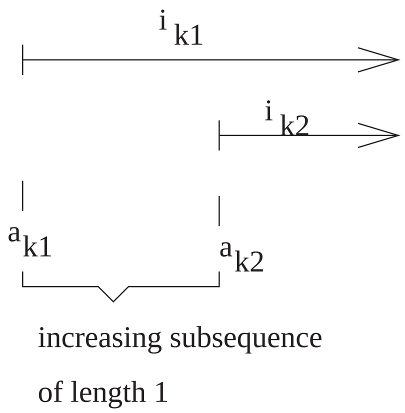

Figure 6: Contradiction from increasing subsequence

From the figure, it can be seen that $i_{k1} \leq i_{k2}+1$, but this is contradiction, since we already have the two i_ks equal.

If $a_{k1} > a_{k2}$, impossible situation arises, see figure 7.

From the figure, it can be seen that $d_{k1} \leq d_{k2}+1$, but this is contradiction, since we already have the two d_ks equal.

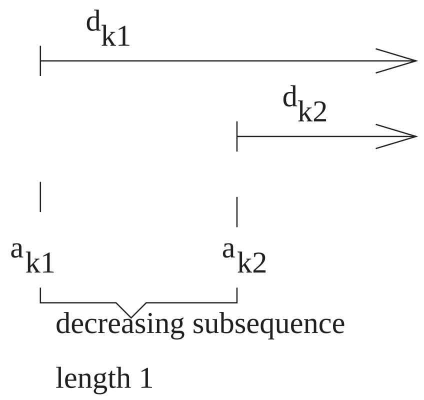

Figure 7: Contradiction from decreasing subsequence

Permutation - a revision

Permutation is an ordered arrangement. An ordered arrangement of r elements of a set is called an r-permutation.

Example: If S=1,2,3, then 3,2 is a 2-permutation of S.

Formula:

The number of r-permutation of a set with n elements:

$P(n,r) = n(n-1)(n-2)\dots(n-r+1) = \frac{n!}{(n-r)!}$

Example: If there are 8 people in a race, how many ways can gold, silver, bronze medals be awarded?

Answer $= P(8,3) = 8*7*6 = 336$.

Combination - a revision

Combination is an unordered arrangement. An alternative name is binomial coefficient.

Formula:

$$C(n,r) = \binom{n}{r} = \frac{n!}{(n-r)!r!}$$

$$P(n,r) = C(n,r) * P(r,r)$$

If $n, r \in Int$ and $r \leq n$,

$$C(n,r) = C(n,n-r)$$

Let n be a positive integer, then

$$\sum_{k=0}^{n} C(n,k) = 2^n$$

Pascal's Identity

Let n and k be positive integers and $n \geq k$, then

$$C(n,k-1) + C(n,k) = C(n+1,k)$$

Prove:

Let a set T have n+1 elements.

Let 'a' be an element of T.

Let S = T - {a }

Number of subsets of T that have k elements = $C(n+1,k)$.

There are two alternative ways of getting these subsets.

- subsets containing 'a', and k-1 elements from S:
$$1 * C(n, k-1)$$
- subsets containing k elements from S (not containing 'a'):
$$C(n, k)$$

Therefore $C(n, k-1) + C(n, k) = C(n+1, k)$.

$\boxed{\text{Vandermonde's Identity}}$

Let m,n,r $\geq$ 0, $(m, n, r \in Int), r \leq m, r \leq n$. Then

$$C(m+n, r) = \sum_{k=0}^{r} C(m, r-k) * C(n, k)$$

Prove:

Let set A have m elements and set B have n elements. The number of ways to pick r elements from $A \cup B$ is $C(m+n, r)$.

There is another way (2 steps) to carry out this task. That is to:

1. pick r-k elements from A:

$$C(m, r-k)$$

2. and then pick k elements from B:

$$C(n, k)$$

So we have : $C(m, r-k) * C(n, k)$

But there are many possibilities for k, so we need to sum up all possibilities. This is why we have Σ in the theorem.

$\boxed{\text{Binomial theorem}}$

Let x and y be variables, and n be a positive integer, then

$$(x+y)^n = \sum_{j=0}^{n} C(n,j)x^{n-j}y^j$$

$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \ldots +$$

$$\binom{n}{n-1} xy^{n-1} + \binom{n}{n} y^n$$

Let n be a positive integer, then

$$\sum_{k=0}^{n} (-1)^k C(n,k) = 0$$

The use of binomial theorem

Finding the coefficient of complex terms.

Example:

1. Find the coefficient of $x^{12}y^{13}$ from $(x + y^{25})$.

   From the theorem, we can directly see that this coefficient is $C(25, 13)$.

2. Find the coefficient of $x^{12}y^{13}$ from $(2x - 3y)^{25}$.

We need to manipulate it a bit so that it is in the form of the formula.

$$[(2x) + (-3y)]^{25} = \sum_{j=0}^{25} C(25, j)(2x)^{25-j}(-3y)^j$$

Therefore the coefficient $= C(25, 13)2^{12}(-3)^{13}$.

Recurrence Relations

A recurrence relation for a sequence a_n is an equation that expresses a_n in terms of one or more previous terms of the sequence.

It is a recursive definition.

A sequence is a solution of a recurrence relation if its terms satisfy the recurrence relation. A solution must be in closed form (not containing any recursive form).

Example: Is $a_n = 3n$ a solution to $a_n = 2a_{n-1} - a_{n-2}, n \geq 2$?

We substitute $a_n = 3n$ in the case of a_{n-1} and a_{n-2}.

Therefore we get:

$$a_n = 2 * 3(n-1) - 3(n-2) = 3n$$

matching what we want. Therefore $a_n = 3n$ is a solution.

The base case, a_0, a_1, etc are called initial conditions.

$\boxed{\text{Writing Recurrence Relations - Examples}}$

1. Deposit 10,000 baht at a bank with interest rate 2 percent. How much will be in the account after 30 years?

   Let a_n be the money in the account after n years.

   We can write it as

   $$a_n = 1.02a_{n-1}, a_0 = 10000$$

   Therefore:

   $$a_{30} = 1.02a_{29} = 1.02^2 a_{28} = \ldots = 1.02^{30} a_0$$

2. The Fibonacci (rabbit) problem.

   A pair (male-female) of super rabbits that cannot die (just born) are placed on an island with no other life form When the pair reaches 2 months old, it produces another pair (male-female) that month and every month after. What is the recurrence relation representing the number of pairs of rabbits at the n^{th} month?

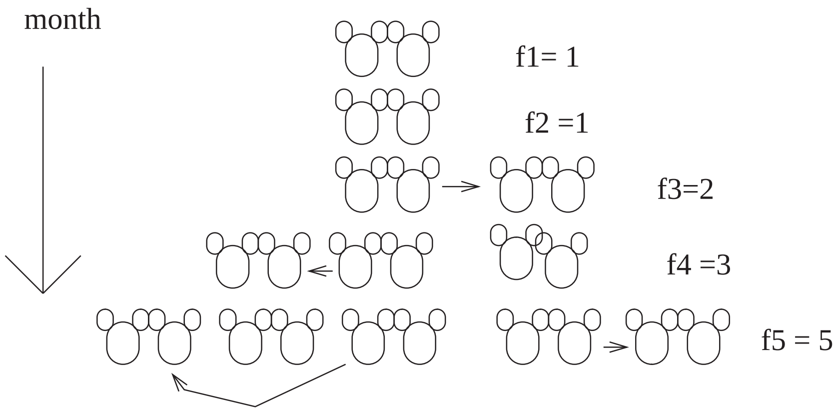

Figure 8: Rabbits from Fibonacci problem

We can observe that

$$f_n = f_{n-1} + f_{n-2}, f_0 = 0, f_1 = 1$$

But to notice is not enough, we must do math induction proof to show that what we notice is correct (later).

3. Tower of Hanoi

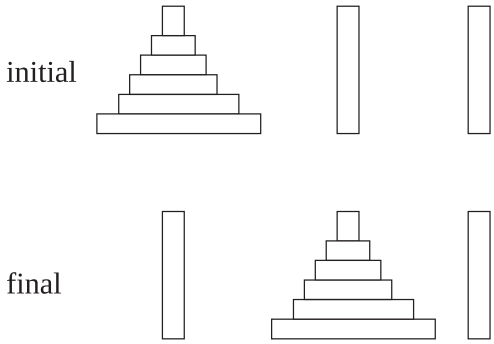

Figure 9: Hanoi

Can move one disk at a time, and a disk must never be placed on top of a smaller disk.

What is the number of moves if there are n disks?

Hanoi Solution

Let a_n be the number of moves for n disks from one pole to another. We can divide the moving of n disks into three steps.

(a) Move $n-1$ disks to a temporary pole. This problem is represented by a_{n-1}, see figure 10.

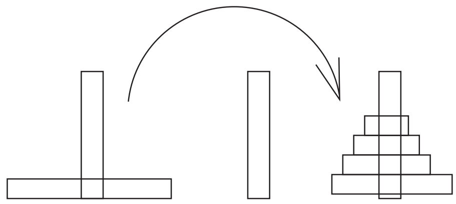

Figure 10: Move n-1 disks to a temporary pole

(b) Move the largest pole to the final pole, this is one move only, see figure 11.

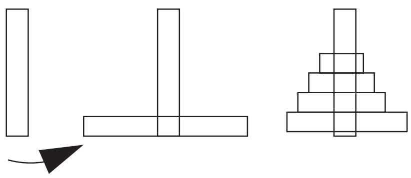

Figure 11: Move the largest disk to the final pole

(c) Move all the disks we moved in the first step to the final pole. This is a_{n-1} again, see figure 12.

$$a_n = 2a_{n-1} + 1, a_1 = 1$$

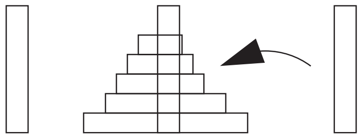

Figure 12: Move the largest disk to the final pole

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To find the closed form (we will see more later), we can distribute each term, starting from a_n:

$$a_n = 2(2a_{n-2} + 1) + 1 = \ldots = 2^n - 1$$

4. Find recurrence relation for the number of bit strings of length n that do not have two consecutive 0s.

   Let a_n be the number of strings of length n that do not have two consecutive 0s.

   $a_n =$ start with 1*a_{n-1} + start with 01 * a_{n-2}

   $= a_{n-1} + a_{n-2}, a_0 = 1, a_1 = 2$

5. Find recurrence relation for laying the floor with 1*2 tile, where the floor size is 2*n

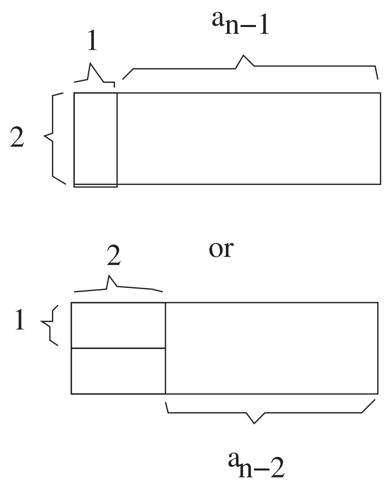

Figure 13: Laying tiles in this case has 2 methods

Therefore the recurrence relation is:

$$a_n = a_{n-1} + a_{n-2}$$

6. If we have $x_0 * x_1 * \ldots * x_n$, how many ways can brackets be put to divide steps of multiplication?

Let a_n be the number of ways to put brackets for $x_0 * x_1 * \ldots * x_n$.

There must be a last multiplication:

$$\underbrace{(x_0 * \ldots * x_k)}_{a_k ways} * \underbrace{(x_{k+1} * \ldots * x_n)}_{a_{n-(k+1)} ways}$$

We now have $a_k * a_{n-k-1}$, but since k can be 0,1,2,... up to n-1.

Therefore

$$a_n = \sum_{k=0}^{n-1} a_k a_{n-k-1}, a_0 = 1$$

7. Find recurrence relation of the number of ternary string of length n that does not contain 00 and 11.

   Let a_n be the number of ternary strings of length n that do not contain 00 and 11.

   Let $a_{n,k}$ be the number of ternary strings of length n that do not contain 00 and 11, and start with k.

   Therefore $a_n = a_{n,0} + a_{n,1} + a_{n,2}$

   We can see that

   - $$a_{n,2} = 1 * a_{n-1}$$

     This means the first element is 2 (1 way to do this). The rest will then be a problem of a_{n-1}.
   - $$a_{n,0} = a_{n-1,1} + a_{n-1,2}$$

     0 is already chosen to be the first element (1 way). Therefore the next element cannot be 0 (by the constraint of this problem). That means it can be either 1 or 2 and the rest is a problem of size $n - 1$.
   - $$a_{n,1} = a_{n-1,0} + a_{n-1,2}$$

- Same thought as the previous case.

$$a_{n-1} = a_{n-1,0} + a_{n-1,1} + a_{n-1,2}$$

- This is common sense from our rule.

$$a_{n-2} = a_{n-1,2}$$

  a_{n-1} that starts with 2 (1 way) is equal to a_{n-2} for this problem.

Therefore

$$a_n = (a_{n-1,1}+a_{n-1,2})+(a_{n-1,0}+a_{n-1,2})+a_{n-1}$$

$$= a_{n-1} + a_{n-1,2} + a_{n-1} = 2a_{n-1} + a_{n-2}$$

$$a_0 = 1, a_1 = 3$$