Divide and Concur recurrence relation

Problem size n is divided into a subproblems of size $\frac{n}{b}$. After solving each all subproblems, we need further $g(n)$ operations to combine those results. Thus the recurrence relation of this kind of problem is:

$$f(n) = a * f(\frac{n}{b}) + g(n)$$

How do we solve this?

1. Change its variable
2. If it is in the form

   $$f(n) = a * f(\frac{n}{b}) + cn^d, n > n_0$$

   we can solve it with a specific method (see later).

Changing Variable

Example: Solve $B_n = 3B_{\frac{n}{2}} + n, B_1 = 1$.

Let $n = 2^k$ and $A_k = B_{2^k}$, therefore $A_{k-1} = B_{2^{k-1}}$ and we can rewrite the original recurrence relation as:

$$A_k = 3A_{k-1} + 2^k \tag{1}$$

Find $A_k^{(h)}$: $r - 3 = 0$, therefore $r = 3$.

Thus $A_k^{(h)} = \alpha * 3^k$

Find $A_k^{(p)}$: $A_k^{(p)} = p * 2^k$, substitute this in equation 1, we get:

$$\begin{aligned} p * 2^k &= 3 * p * 2^{k-1} + 2^k \\ -2 &= p \end{aligned}$$

Therefore

$$A_k^{(p)} = -2 * 2^k = -2^{k+1}$$

$$A_k = \alpha * 3^k + -2^{k+1}$$

Since $1 = B_1 = B_{2^0} = A_0$, We have $1 = \alpha * 3^0 + -2^{0+1}$. Solving this, we get $\alpha = 3$.

Therefore

$$A_k = 3^{k+1} - 2^{k+1}$$

We have to transform it back in term of B. We know $k = log_2 n$.

$$\begin{aligned} B_{2^k} &= 3^{k+1} - 2^{k+1} \\ B_n &= 3^{log_2 n+1} - 2^{log_2 n+1} \end{aligned}$$

Solving $f(n) = a * f(\frac{n}{b}) + cn^d$

We try to find formula.
First, we know:

- $a \geq 1$: one big problem must surely consists of more than one smaller problems.
- $c > 0$: the combination of results must take some time.
  - if $c = 0$, then $cn^d = 0$, which is impossible.
  - if $c < 0$, then $cn^d < 0$, which is impossible.
- $d \geq 0$: if $d < 0$, cn^d will decrease when n increases. This is impossible because if the problem gets larger, it should take more time to combine the results of subproblems.
- $b > 1$: this is because $\frac{n}{b}$ must $< n$.
- $n = b^i n_0$: just to make n divisible by b.

Substitute $n = b^i n_0$ in $f(n) = a * f(\frac{n}{b}) + cn^d$:

$$f(b^i n_0) = a * f(\frac{b^i n_0}{b}) + c(b^i n_0)^d$$

Let $h_i = f(b^i n_0)$. From the above equation, we get:

$$h_i = a * h_{i-1} + cn_0^d * (b^d)^i \qquad (2)$$

This is now in the form of non-homogeneous recurrence relation.

Now we find $h_i^{(h)}$:

The characteristic equation is $0 = r - a$, therefore $r = a$ and

$$h_i^{(h)} = \alpha * a^i$$

Now we find $h_i^{(p)}$. There are two possible values for this, where $a \neq b^d$ and $a = b^d$.

$$\boxed{\text{When } a \neq b^d.}$$

$$h_i^{(p)} = p * (b^d)^i$$

Substitute this in equation 2, we get:

$$\begin{aligned} p * (b^d)^i &= a * p * (b^d)^{i-1} + cn_0^d * (b^d)^i \\ p &= \frac{a * p}{b^d} + cn_0^d \\ (1 - \frac{a}{b^d})p &= cn_0^d \\ p &= \frac{cn_0^d}{(1 - \frac{a}{b^d})} \end{aligned}$$

Therefore

$$h_i^{(p)} = p * (b^d)^i = \frac{cn_0^d}{(1 - \frac{a}{b^d})} * (b^d)^i$$

Now we combine $h_i^{(h)}$ and $h_i^{(p)}$:

$$h_i = \alpha * a^i + \frac{cn_0^d}{(1 - \frac{a}{b^d})} * (b^d)^i$$

Since $n = b^i n_0$, we know $i = log_b(\frac{n}{n_0})$, we can transform h_i back to $f(n)$:

$$\begin{aligned} f(n) &= \alpha * a^{log_b(\frac{n}{n_0})} + \frac{cn_0^d}{(1 - \frac{a}{b^d})} * (b^i)^d \\ &= \alpha * (\frac{n}{n_0})^{log_b a} + \frac{cn_0^d}{(1 - \frac{a}{b^d})} * (\frac{n}{n_0})^d \\ &= c_1 n^{log_b a} + (\frac{c}{1 - \frac{a}{b^d}}) * n^d \end{aligned}$$

where $c_1 = \frac{\alpha}{n_0^{log_b a}}$.

$\boxed{\text{When } a = b^d.}$

$$h_i^{(p)} = p * (b^d)^i * i$$

Doing it in the same way, we will finally get

$$h_i^{(p)} = cn_0^d * i * (b^d)^i$$

and

$$f(n) = c_1 n^d + cn^d log_b(\frac{n}{n_0})$$