Strong Induction (Second Principle)

Example:

There are two piles of cards, players take turn:

- each turn: one player removes any number of cards from 1 pile (any of the two).
- The player who removes the last card wins.

Show that if, in the beginning, the two piles contain the same number of cards, then the second player can always win.

Answer:

Let n be the number of cards in each pile.

1. **Base case:** When n=1, the first player can only remove 1 card from 1 pile. There is no other choice for him. So the second player can remove the 1 remaining card in another pile and win.

2. **Inductive Hypothesis:** For $n = 1$ to k, the second player always wins.

3. **Proof:** We must show that second player can win when $n = k + 1$.

   When $n = k + 1$, let us say the first player removes j cards from one pile, leaving $k + 1 - j$ cards in the pile.

   So the second player can remove j cards from the other pile, leaving the same amount on both pile.

   And it can be seen that $1 \leq k+1-j \leq k$, hence the second player can win by the inductive hypothesis.

Example: Prove that the number 12 or more can be formed by adding multiples of 4 and/or 5.

Answer:

Let n be the number we are interested in.

We first use Normal Induction:

1. **Base case:** $n = 12$, this can be formed from $4+4+4$. Thus base case proven.
2. **Inductive Hypothesis:** For $n = k$, n is multiples of 4 and/or 5.
3. **Proof:** We must show that $k+1$ is multiples of 4 and/or 5.

   for $k+1$

   - If at least one 4 is used for the case where $n = k$, then replace this 4 by 5, and we therefore get $n = k+1$ from the additions of 4 and/or 5.
   - If there is no 4, it means only 5 are used. Since $k \geq 12$, at least three 5 are used. Replace this $5+5+5$

by $4 + 4 + 4 + 4$ and we will get $k + 1$, which is still the additions of 4 and/or 5.

Now we will proof this using strong induction.

1. **Base case:** is divided into the following cases:
   - when we have 12: This is $4 + 4 + 4$.
   - when we have 13: This is $4 + 4 + 5$.
   - when we have 14: This is $5 + 5 + 4$.
   - when we have 15: This is $5 + 5 + 5$.

   These are the proven base cases.
2. **Inductive Hypothesis:** Let $k \geq 15$. Assume that all numbers from 12 to k is the result of adding 4 and/or 5.
3. **Proof:** We must show that $k + 1$ is multiples of 4 and/or 5.

   for $k + 1$
   - We use the result of $k - 3$, which

satisfies the hypothesis, and add it by 4. This completes the proof.

- Note that we have many subcases for base case because the induction by using $k-3$ does not work with $13, 14$, and 15.

Well-Ordering Property

Every nonempty set of nonnegative integers has a least element.

We can use this directly in proof.

Example: Prove that, if a is an integer and d is a positive integer, then there are unique integers q and r with $0 \leq r < d$ and $a = dq + r$.

Answer:

First, do the existence proof;

OK. Let $S = \{(a-dq) \in Int | (a-dq) \geq 0\}$. By the well-ordering property, S has a least element, $r = a - dq_0$.

This is assuming there exists $a = dq + r$. Then we see if, at the same time, $r < d$ is true. If true, then this is the existence example we want.

$0 \leq r$ because r must be a member of S, as defined by us.

We show that $r < d$ by assuming the negation is true:

$$
\begin{aligned}
r &\geq d \\
r - d &\geq 0 \\
(a - dq_0) - d &\geq 0 \\
a - d(q_0 + 1) &\geq 0
\end{aligned}
$$

This means $a - d(q_0 + 1)$ must also be a member of S, and this is less than r. Contradiction, because r is already the smallest number in this set.

Therefore $r < d$. Now we have the existence proof.

The second step is to prove the uniqueness.

Do the contradiction proof:

1. If q is not unique that means there exists

$$q2 \neq q$$

, a= dq+r = dq$_2 + r$

We can derive further that:

$$dq + r = dq_2 + r$$

$$dq = dq_2$$
$$q = q_2$$

Contradiction, q must not be equal to q_2. So we conclude that q is unique.

2. If r is not unique... this is exactly the same method of proof. We will get the uniqueness result.

Example: In one tournament football match of m teams ($m \geq 3$). Each team plays every other team once. Prove that if there is a cycle, for example "A beats B beats C ...", then there is a cycle of exactly three teams. (There is no draw game)

Answer:

Prove by contradiction. Assume there is no cycle of length 3.

Let S be a set of all cycle length (S is not empty since we know there is a cycle for sure). By well-ordering property, let k be the least element of S.

From our assumption, $k > 3$.

Let's look at one cycle "A beats B beats C beats ... " of length k. What would the result of A and C be?:

- If C beats A, then there is a cycle of length 3. Contradiction.
- If A beats C, then we can form a new cycle by omitting B from the cycle we are looking at. Thus obtaining a cycle of length $k-1$. But this is contradiction, since k is already the smallest cycle size.

Infinite Descent Proof

is to show that for a proposition $P(n)$, $P(k)$ is false for all positive integer k.

1. Assume that $P(k)$ is true for at least 1 k.
2. By the well-ordering property, there is the least positive integer j such that $P(j)$ holds.

3. Find j' such that $j' < j$ and $P(j')$ is true. Thus contradiction.

This is what Fermat used to prove there is no solution to

$$x^4 + y^4 = z^4$$

Example: Prove that $\sqrt{2}$ is irrational.
Answer:

1. Assume $\sqrt{2}$ is rational then $\sqrt{2} = \frac{m}{n}$.
2. By the well-ordering property, there is a least positive integer N, such that $\sqrt{2} = \frac{M}{N}$. And we know $M^2 = 2N^2$
3. Now we do this:

$$\begin{aligned}\frac{M}{N} &= \frac{(M-N)M}{(M-N)N} \\ &= \frac{M^2 - MN}{(M-N)N} \\ &= \frac{2N^2 - MN}{(M-N)N}\end{aligned}$$

$$= \frac{(2N - M)N}{(M - N)N}$$
$$= \frac{2N - M}{M - N}$$

We now have $M - N$ as a denominator.

4. Because $1 < \sqrt{2} < 2$:

$$1 < \sqrt{2} < 2$$
$$1 < \frac{M}{N} < 2$$
$$N < M < 2N$$
$$0 < M - N < N$$

Therefore $M - N < N$, contradicting our original assumption.

$\boxed{\text{Structural Induction}}$

Example: Given the following string length definition, where w is a string and x is a character:

$$l(emptyString) = 0,$$

$$l(wx) = l(w) + 1$$

Prove that:

$$l(xy) = l(x) + l(y)$$

where x and y are in σ^*, the set of strings over alphabet σ.

Answer:

1. **Base case:** We must show that

   $$l(x{*}emptyString) = l(x){+}l(emptyString)$$

   for all $x \in \sigma^*$.

   $$l(x{*}emptyString) = l(x) = l(x){+}l(emptyString)$$

   It is obvious that the base case is true.

2. **Recursive step:**

Assume $l(xy) = l(x) + l(y)$.
We must show that $l(xya) = l(x) + l(ya)$ for every $a \in \sigma$.
From the original definition:

$$l(xya) = l(xy) + 1$$

and

$$l(ya) = l(y) + 1$$

Therefore, with the assumption, we get:

$$l(xya) = l(xy) + 1 = (l(x) + l(y)) + 1 = l(x) + (l(y) + 1) = l(x) + l(ya)$$

Example: Given the definitions of full binary tree as follows:

height, for a full binary tree, T:

- $h(T) = 0$, if T consists of only a root.
- $h(T) = 1 + max(h(T1), h(T2))$, where $T1$ and $T2$ are full binary trees.

number of nodes for a full binary tree, T:

- $n(T) = 1$, if T consists of only a root.
- $n(T) = 1 + n(T1) + n(T2)$, where $T1$ and $T2$ are full binary trees.

Prove that for a full binary tree, T, $n(T) \leq 2^{h(T)+1} - 1$.

Answer:

1. **Base case:**

$$n(onlyRoot) = 1 \leq 2^{0+1} - 1$$

   Thus the base case holds.

2. **Inductive step:** Assume that:

   - $n(T1) \leq 2^{h(T1)+1} - 1$

- $n(T2) \leq 2^{h(T2)+1} - 1$

3. **Prove:**

$$
\begin{aligned}
n(T) &= 1 + n(T1) + n(T2) \\
&\leq 1 + (2^{h(T1)+1} - 1) + (2^{h(T2)+1} - 1) \\
&\leq 2 * max(2^{h(T1)+1}, 2^{h(T2)+1}) - 1 \\
&\leq 2 * 2^{max(h(T1),h(T2))+1} - 1 \\
&\leq 2 * 2^{h(T)} - 1 \\
&\leq 2^{h(T)+1} - 1
\end{aligned}
$$

Thus the proof completes.