Logic and connectives

- Proposition == statement which is true or false, but not both.
- Let p and q be a proposition:
  - $p \wedge q$ is true only when both p and q are true, and false otherwise
  - $p \vee q$ is true when at least one of p and q is true
  - $p \oplus q$ is true when exactly one of p and q is true, and false otherwise.
  - $p \rightarrow q$ is false only when p is true and q is false.
  - $p \leftrightarrow q$ is true when p and q have the same truth value

More on $p \rightarrow q$

In English

- if p then q
- p implies q
- if p, q
- p only if q
- p is sufficient for q
- q if p
- q whenever p
- q is necessary for p

p unless q is $\neg q \rightarrow p$ or sometimes $p \leftrightarrow q$

Converse of $p \rightarrow q$ is $q \rightarrow p$

contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$

Translation example

If Jojo is over 21 and either he has previously been sentenced to imprisonment or non-imprisonment is not appropriate for him then a custodial sentence is possible:

$$j \wedge (i \vee \neg a) \rightarrow c$$

$$\begin{aligned}
&(over21(Jojo) \\
&\wedge (previousprisonment(Jojo) \vee \neg nonprisonOK(Jojo))) \\
&\rightarrow possiblecusto(Jojo)
\end{aligned}$$

Summary so far

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$	$p \leftrightarrow q$
T	**T**	**F**	**T**	**T**	**F**	**T**	**T**
T	**F**	**F**	**F**	**T**	**T**	**F**	**F**
F	**T**	**T**	**F**	**T**	**T**	**T**	**F**
F	**F**	**T**	**F**	**F**	**F**	**T**	**T**

Equivalence

- Tautology == proposition that is always true, no matter what truth values each proposition has i.e. $a \vee True$ is true.
- Contradiction == proposition that is always false. i.e. $a \wedge False$
- Contingency == neither Tautology nor Contradiction.

$\Leftrightarrow, \equiv$

- P is logically equivalent to q if $p \leftrightarrow q$ is a tautology
- We show equivalence (primarily) by truth table, example:

$$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$$

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	**T**	**F**	**F**	**T**	**F**	**F**
T	**F**	**F**	**T**	**T**	**F**	**F**
F	**T**	**T**	**F**	**T**	**F**	**F**
F	**F**	**T**	**T**	**F**	**T**	**T**

$$p \wedge True \Leftrightarrow p$$

$$p \vee False \Leftrightarrow p$$

Identity law

$$p \vee True \Leftrightarrow True$$

$$p \wedge False \Leftrightarrow False$$

Domination laws

$$p \vee p \Leftrightarrow p$$

$$p \wedge p \Leftrightarrow p$$

Idempotent laws

$$\neg(\neg p) \Leftrightarrow p$$

Double negation

$$p \vee q \Leftrightarrow q \vee p$$

$$p \wedge q \Leftrightarrow q \wedge p$$

Commutative laws

$$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$$

$$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$$

Associative laws

$$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$$

$$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$$

Distributive laws

De Morgan's laws

$$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$$

$$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$$

$$p \rightarrow q \Leftrightarrow \neg p \vee q \Leftrightarrow \neg(p \wedge \neg q)$$

$$p \leftrightarrow p \Leftrightarrow p$$

$$p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$$

$$p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$$

$$p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$$

Examples

$$(p \wedge q) \to (p \vee q) \text{.......} \textit{always..true}\,?$$

$$\neg(p \vee (\neg p \wedge q)) \Leftrightarrow \neg p \wedge \neg q\,?$$

$$p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)?$$

Predicates

- x>3 ---------> greaterthan3(x)

  Predicate tells

  us what x is

- Predicate can have more than 1 variable i.e. Q(x,y,z,...)

  tuple

Quantifier

- Universal (for all) $\forall$
  - $\forall x$ greaterthan3(x)
  - $\forall x$ (inthisclass(x) $\rightarrow$ done_calculus(x))

    There must be an actual range in real use

  - [(inthisclass(a) $\rightarrow$ done_calculus(a)] $\wedge$ [(inthisclass(b) $\rightarrow$ done_calculus(b)]

    $\wedge$ …

Quantifier (cont.)

- Existential (there exists) $\exists$
  - $\exists$ x (inthisclass(x) $\rightarrow$ done_calculus(x))
  - [(inthisclass(a) $\rightarrow$ done_calculus(a)] $\vee$ [(inthisclass(b) $\rightarrow$ done_calculus(b)] $\vee$ …

Examples

$$\exists x[striped(x) \wedge hungry(x)]$$

Same x   i.e. If x is “mycat” then “mycat” must apply to both predicates.

$$\exists x(striped(x)) \wedge \exists x(hungry(x))$$

Two x here are independent

This is the same for $\forall$

Order of Quantifiers

$\exists x \exists y \exists z \ldots$

$\forall x \forall y \forall z \ldots$

Can be arranged in any order

$\forall x \exists y$

$\exists x \forall y$

Must maintain original order

$\forall x \exists y [mother(y, x)]$ Everyone has a mother

$\exists y \forall x [mother(y, x)]$ There is a single person who is a mother of everyone

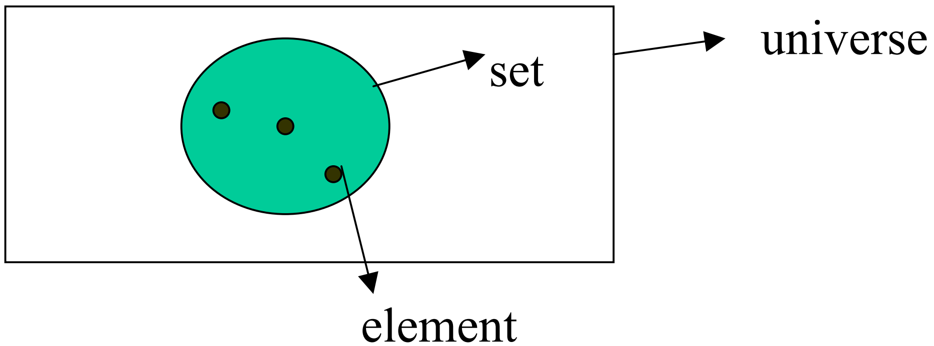
Negation Properties of quantifier

$$\neg\forall x \Leftrightarrow \exists x \neg$$

$$\neg\exists x \Leftrightarrow \forall x \neg$$

Set

- O = {1,3,5,7,9}, P = {2,r,4,h,n}
- Two sets are equal (=) if they have the same elements
  - {1,3,5} = {3,5,1,1,5}
- <u>Set builder notation</u>: i.e. stating the properties of the set elements
  - O = {x | x is an odd positive integer less than 10}
- <u>Venn Diagram</u>



Set terms and notations

$a \in A$ A is an element in A

$a \notin A$

ϕ { } Empty set has no element

$A \subseteq B$ Every element of A is also in B

$A \subset B$ Proper subset

$A = B$ when $(A \subseteq B) \wedge (B \subseteq A)$

Empty set is a subset of every set

a set is always a subset of itself

Set terms and notations (cont.)

If S has n distinct elements S is a finite set. n is the <u>cardinality</u> of S

$$|S|$$

Powerset = set of all subsets

Example: P({0,1,2}) = { ϕ,{0},{1},{2},{0,1},{0,2},{1,2},{0,1,2}}

$$P(\phi) = \{\phi\}$$

$$P(\{\phi\}) = \{\phi, \{\phi\}\}$$

Cartesian Product

- Sets do not order elements, but **<u>tuples</u>** do.
  - (a1,a2,a3,a4)
  - tuples of 2 elements are called ordered pairs
- Cartesian product of set A and B

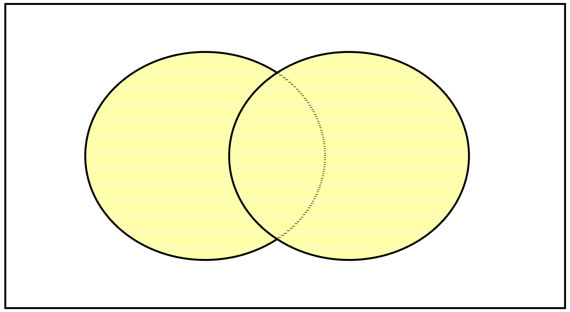
$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

  - A = {1,2}, B = {a,b,c}   AxB = {(1,a),(1,b),(1,c),(2,a),(2,b),(2,c)}
- Cartesian product of n sets

$$A_1 \times A_2 \times A_3 \times \ldots A_n = \{(a_1, a_1, a_1, \ldots, a_n) \mid a_i \in A_i\}$$ Where i=1,2,…n

Set Operations

- Union



$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$$

  - e.g $\{1,3,5\} \cup \{1,2,3\} = \{1,2,3,5\}$
  - Note: $|A \cup B| = |A| + |B| - |A \cap B|$

- Complement

$$\overline{A} = U - A = \{x \mid x \notin A\}$$

- Intersection

$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$$

  - e.g. $\{1,3,5\} \cap \{1,2,3\} = \{1,3\}$
  - Two sets, A and B, are "disjointed" if $A \cap B = \phi$

- Difference

$$A - B = \{x \mid (x \in A) \wedge (x \notin B)\}$$

  - it is a complement of B with respect to A
  - e.g. $\{1,3,5\} - \{1,2,3\} = \{5\}$

    $\{1,2,3\} - \{1,3,5\} = \{2\}$

$$A \cup \phi = A$$

$$A \cap U = A$$

Identity law

$$A \cup U = U$$

$$A \cap \phi = \phi$$

Domination laws

$$A \cup A = A$$

$$A \cap A = A$$

Idempotent laws

$$\overline{\overline{A}} = A$$

Complementation law

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

Commutative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Associative laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Distributive laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

De Morgan's laws

To show 2 sets are equal

- Show that one is a subset of the other and vice versa.
- Use set builder notation and logic
  - e.g. $\overline{A \cap B} = \overline{A} \cup \overline{B}$???

$$
\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} \\
&= \{x \mid \neg(x \in A \cap B)\} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} \\
&= \{x \mid x \notin A \vee x \notin B\} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\}
\end{aligned}
$$

To show 2 sets are equal (cont.)

- Use membership table- similar to truth table

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

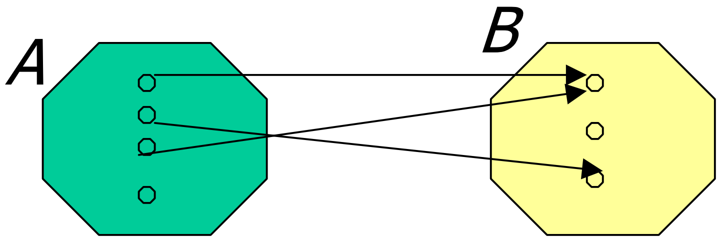
A	B	$\overline{A}$	$\overline{B}$	$A \cap B$	$\overline{A \cap B}$	$\overline{A} \cup \overline{B}$
T	**T**	**F**	**F**	**T**	**F**	**F**
T	**F**	**F**	**T**	**F**	**T**	**T**
F	**T**	**T**	**F**	**F**	**T**	**T**
F	**F**	**T**	**T**	**F**	**T**	**T**

Other useful set terms

- Bit string- representing set on computer
  - if U = {1,2,3,…,10} then odd integers in U can be represented by:
  - 1010101010
- Symmetric difference- this is just like exclusive or $A \oplus B$
- Successor of A is $A \cup \{A\}$
- Multisets - one element can occur more than once
  - {1,1,1,2,2,3,3}}   In this case 3 is the multiplicity of element "1"

Functions

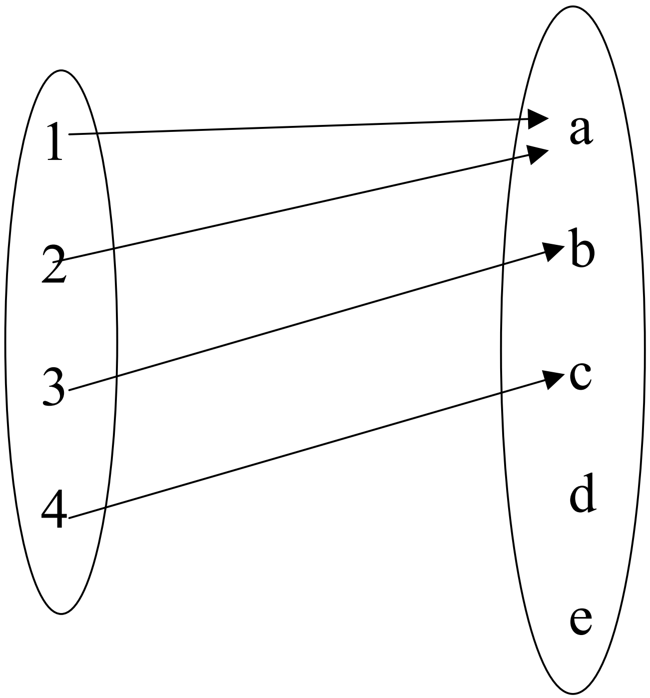
- Let A and B be sets. A function from A to B is



$$f : A \to B$$
$$f(a) = b$$

- A = domain of f, B = codomain of f, a = pre-image of b, b = image of a, range of f = set of all images of elements of A

Function Example



Domain ={1,2,3,4}

codomain = {a,b,c,d,e,}

range = {a,b,c}

image of subset S ={1,2,3}

is {a,b}

One-to-one (injective) function

$$f(x) = f(y) \to x = y$$

- Strictly increasing function (this must be 1-to-1)
  - if f has domain and codomain as subset of real numbers,
  - f is strictly increasing if $f(x) < f(y)$ whenever $x<y$ (x,y are in the domain of f)
  - f is strictly decreasing if $f(x) > f(y)$ whenever $x<y$ (x,y are in the domain of f)

Onto (surjective) and bijection

- Onto function has codomain = range
  - e.g. f(x)=x*x from set of integers to set of integers is not onto
- bijection or 1-to-1 correspondence is both 1-to-1 and onto
  - e.g. identity function is 1-to-1 and onto

Inverse and composition

- Function f must be 1-to-1 and onto in order to have inverse.
  - Not 1-to-1 : the inverse won't be a function
  - not onto: there is a b that can't map back to a
- if $[(g : A \rightarrow B) \wedge (f : B \rightarrow C)] \rightarrow$
  $(f \circ g)(a) = f(g(a))$
  - e.g. if A= {a,b,c} and B= {1,2,3}, let $(g : A \rightarrow A), (f : A \rightarrow B)$
  - g(a) = b, g(b) = c, g(c) = a, f(a) =3, f(b) = 2, f(c) = 1….

$$(f \circ g)(a) = f(g(a)) = f(b) = 2 ... etc$$

$$(g \circ f)(a) = g(f(a)) = g(3) = undefined$$

$$f \circ g \neq g \circ f$$

- E.g. if both f and g are functions from/to integers
  - f(x) = 2x+3   g(x) = 3x+2
  - (f o g)(x) = f(g(x)) = f(3x+2) = 2(3x+2)+3= 6x+7
  - (g o f)(x) = g(f(x)) = g(2x+3) = 3(2x+3) +2 = 6x+11
- Note: $f \circ f^{-1}, f^{-1} \circ f$ are identity functions

Graph of function

- Let f be a function from set A to set B.
  - The graph of f is the set: $\{(a,b) \mid a \in A \wedge f(a) = b\}$
  - It is a subset of the cartesian product of A and B
  - plot it on x,y coordinate

Floor and Ceiling functions

- Let x be a real number. A floor function rounds x down to the closest integer <= x. A ceiling function rounds x up to the closest integer >=x

$$\lceil x \rceil = 6$$

$$x = 5.56$$

$$\lfloor x \rfloor = 5$$

Properties of floor and ceiling

$$x - 1 < \lfloor x \rfloor \le x < \lfloor x \rfloor + 1$$

$$\lceil x \rceil - 1 < x \le \lceil x \rceil < x + 1$$

$$\lfloor -x \rfloor = -\lceil x \rceil$$

$$\lceil -x \rceil = -\lfloor x \rfloor$$

$$\lfloor x + \text{int} \rfloor = \lfloor x \rfloor + \text{int}$$

$$\lceil x + \text{int} \rceil = \lceil x \rceil + \text{int}$$

Sequence

- A sequence is a function from a subset if INT to a set S
  - a_n is the image of integer n. It is a "term" of the sequence
  - if a_n =1/n then a sequence starting with a_1 is 1,1/2,1/3,1/4,…
  - finite sequence is called "string"
    - length of string is the number of terms in that string
- Arithmetic progression sequence is a sequence that has the form:

$$a, a+d, a+2d, a+3d, \ldots, a+nd$$

- Geometric progression:

$$a, ar, ar^2, ar^3, \ldots, ar^k$$

Summations

$$a_m + a_{m+1} + \ldots + a_n = \sum_{j=m}^{n} a_j$$

Upper limit

Index of summation

Lower limit

- To change index, example: from 1 to 5 to 0 to 4, just let k = j-1

$$\sum_{j=1}^{5} j^2 = \sum_{k=0}^{4} (k+1)^2$$

- For nested summation, do the inner summation first
- To find general formula, example: (see next page)

$$S = \sum_{j=0}^{n} ar^{j}$$

$$rS = r\sum_{j=0}^{n} ar^{j} = \sum_{j=0}^{n} ar^{j+1}$$

$$rS = \sum_{k=1}^{n+1} ar^{k} = \sum_{k=0}^{n} ar^{k} + (ar^{n+1} - a)$$

$$rS = S + (ar^{n+1} - a)$$

$$S = \frac{ar^{n+1} - a}{r - 1}$$

Useful Summation (can be inductively proven)

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Function and Cardinality

- Sets A and B have the same cardinality iff there is a one-to-one correspondence from A to B
- A set that is finite or has the same cardinality as the set of natural number is *countable*. e.g. odd positive integers
  - Another way to think : *countable* iff we can list that set' elements in a sequence
- A set that is not countable is *uncountable* e.g. a set of real numbers

Growth of function (Big O)

- Let f and g be functions from integers or real to real. f(x) is O(g(x)) if $|f(x)| \le C|g(x)|$ whenever x>k -> C,k are constants
  - the pair C and k is never unique
  - e.g. $x^2 + 2x + 1$ is $O(x^2)$
  - because $x^2 + 2x + 1 \le x^2 + 2x^2 + x^2 = 4x^2$
  - notice that x^2 is also $O(x^2 + 2x + 1)$ thus the two functions are of the same order
- g(x) can be replaced by a function with larger absolute value

Theorem of Big O

- Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ where all a_n are real numbers. Then $f(x) = O(x^n)$
- Big O therefore can estimate function e.g.:
  - 1+2+3+…+n <= n+n+n+…+n = n^2
  - n! = 1*2*3*…*n <= n*n*...*n = n^n
  - and thus $\log n! \leq \log n^n = n \log n = O(n \log n)$

Growth of combinations of functions

- To find big O of f1+f2:

$$\left|(f_1 + f_2)(x)\right| = \left|f_1(x) + f_2(x)\right|$$
$$\leq \left|f_1(x)\right| + \left|f_2(x)\right|$$
$$< C_1\left|g_1(x)\right| + C_2\left|g_2(x)\right|$$
$$\leq C_1 \max(\left|g_1(x)\right|, \left|g_2(x)\right|) + C_2 \max(\left|g_1(x)\right|, \left|g_2(x)\right|)$$
$$= (C_1 + C_2)\max(\left|g_1(x)\right|, \left|g_2(x)\right|)$$

- Therefore, $(f_1 + f_2)(x) = O(\max(g_1(x), g_2(x)))$
  - x > max(k1,k2)

Growth of combinations of functions (cont.)

- What about f1*f2

$$\left|(f_1 f_2)(x)\right| \le C_1\left|g_1(x)\right|C_2\left|g_2(x)\right|$$
$$\le (C_1 C_2)\left|g_1 g_2(x)\right|$$

- Therefore $(f_1 f_2)(x) = O(g_1(x)g_2(x))$

The use of Big O: example

- Use Big O to estimate

$$f(x) = 3x\log(x!) + (x^2+3)\log x$$

$$x \qquad \underbrace{x^x}_{x\log x} \qquad x^2$$

$$O(x^2 \log x)$$

Big Omega and Big Theta

- Big O is only the upper bound
- a lower bound is Big Omega. Theta indicates both lower bound and upper bound.
- Let f and g be functions from integers or real to real:
  - f(x) is $\Omega(g(x))$ if $|f(x)| \geq C|g(x)|$ whenever x>k -> C,k are constants
  - f(x) is $\Theta(g(x))$ if $(f(x) = O(g(x))) \wedge (f(x) = \Omega(g(x)))$

Big Omega and Big Theta (examples)

- $f(x) = 8x^3 + 5x^2 + 7 \geq 8x^3$ for all positive real numbers x.

  Thus it is of $\Omega(x^3)$ (it is also $O(x^3)$)

- $f(x) = 1 + 2 + 3 + \ldots + x$ (known to be $O(x^2)$) is also $\Omega(x^2)$

  and therefore $\Theta(x^2)$ because

$$
\begin{aligned}
1 + 2 + 3 + \ldots + x &\geq \frac{x}{2} + (\frac{x}{2} + 1) + \ldots + x \\
&\geq \frac{x}{2} + \frac{x}{2} + \ldots + \frac{x}{2} \\
&\geq \frac{x}{2} * \frac{x}{2} \\
&\geq \frac{x^2}{4}
\end{aligned}
$$