$\boxed{\text{Solving Recurrence Relations}}$

1. Guess and Math Induction

   Example:

   Find the solution for $a_n = 2a_{n-1} + 1, a_0 = 0$

   We can try finding each a_n:

   - $a_0 = 0$
   - $a_1 = 2 * 0 + 1 = 1$
   - $a_2 = 2 * 1 + 1 = 3$
   - $a_3 = 2 * 3 + 1 = 7$
   - $a_4 = 2 * 7 + 1 = 15$

   Observing the result, we see that the result is $a_n = 2^n - 1$.

   But do not answer yet. We need to prove it first, using math induction.

   Base case: $a_0 = 2^0 - 1 = 0$, same result as the given definition.

   Assume: $a_n = 2^n - 1$

   Prove: $a_{n+1} = 2^{n+1} - 1$

   $a_{n+1} = 2a_n + 1$, from recurrence definition.

   $= 2(2^n - 1) + 1$, from the assumption.

   $= 2^{n+1} - 1$, as we wanted.

2. Expand it

$$a_n = 2(2a_{n-2} + 1) + 1 = \ldots = 2^n - 1$$

as we've seen. But this is error-prone.

3. Change variables

Example: 1

$a_n = \frac{(a_{n-1})^2}{a_{n-2}}, a_0 = 1, a_1 = 2, (n > 1)$

Divide both sides by a_{n-1}, we will get

$$\frac{a_n}{a_{n-1}} = \frac{a_{n-1}}{a_{n-2}}$$

So, let $b_n = \frac{a_n}{a_{n-1}}$, the above equation will become:

$$b_n = b_{n-1}$$

And since $\frac{a_1}{a_0} = 2$, 2 will be the value of the first b_n, and all other following b_ns. Therefore

$$a_n = 2a_{n-1}$$

Now that we achieve the simple form, we can use previous methods to find the (in this case is 2^n) solution.

Example: 2

$a_n = n * a_{n-1} + n!, a_0 = 2$

Divide all by $n!$. We will get

$$\frac{a_n}{n!} = \frac{a_{n-1}}{(n-1)!} + 1$$

Let $b_n = \frac{a_n}{n!}$. We will get

$$b_n = b_{n-1} + 1, b_0 = 2$$

Now we can easily find the solution of b_n (we get $b_n = 2 + n$).

But we are not finished. Don't forget the question asks for a_n, not b_n. We can use b_n to find a_n anyway.

$$\frac{a_n}{n!} = b_n = 2 + n$$

Therefore $a_n = n!(2 + n)$.

$\boxed{\text{Solving Recurrence Relation with Formula}}$

Definition 1

A linear homogeneous recurrence relation of degree k with constant coefficients is

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

where $c_1, c_2, \ldots, c_k$ are called the coefficients, are REAL, and $c_k \neq 0$.

Example:

- $a_n = a_{n-1} + (a_{n-2})^2$. Not linear because it has $-^2$ on a_{n-2}.
- $H_n = 2H_{n-1} + 1$. Not homogeneous. Homogeneous contains recursive terms only.
- $B_n = n * B_{n-1}$. Not have constant coefficient.

$\boxed{\text{Deriving the formula}}$

Let the solution of

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

be

$$a_n = r^n$$

where r is constant. Substitute it into the original equation, we get:

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \ldots + c_k r^{n-k}$$

When we have $n = k$ This can be arranged into the form:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_{k-1} r - c_k = 0 \quad (1)$$

This is what we called the Characteristic Equation.

Theorem 1

Let c_1 and c_2 be REAL numbers. Suppose $r^2 - c_1 r - c_2 = 0$ has two distinct roots (say, r_1 and r_2), then

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} \tag{2}$$

has

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \tag{3}$$

as its general solution, where $n = 0, 1, 2, \ldots$ and α_1, α_2 are constants.

Prove: by substituting the formula and see if it remains true.

First we know $r^2 - c_1 r - c_2 = 0$ can be rearranged as

$$r^2 = c_1 r + c_2 = c_1 r_1 + c_2 = c_1 r_2 + c_2 \tag{4}$$

Now we look at a_n (originally from equation 2):

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

If we apply equation 3 to a_{n-1} and a_{n-2}, we get:

$$a_n = c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2})$$

which can simply be rearranged as

$$a_n = \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2)$$

We can substitute r_1^2 and r_2^2 for $c_1 r_1 + c_2$ and $c_1 r_2 + c_2$ respectively (see equation 4). Hence we get:

$$a_n = \alpha_1 r_1^{n-2}(r_1^2) + \alpha_2 r_2^{n-2}(r_2^2)$$

The terms cancelled out, and we are left with

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

which is the solution we want.

Example: 1

Find the solution of $a_n = a_{n-1} + 2a_{n-2}, a_0 = 2, a_1 = 7$.

Answer:

The characteristic equation is $r^2 - r - 2 = 0$.

We can find 2 roots, 2 and -1.

Then we can apply the formula and get

$$a_n = \alpha_1 2^n + \alpha_2(-1)^n$$

Substitute $n = 0$ and 1, and considering the given initial conditions, we get two equations:

$$a_0 = \alpha_1 + \alpha_2 = 2$$

and

$$a_1 = \alpha_1 * 2 + \alpha_2 * (-1) = 7$$

From these two equations, we can find α_1 and α_2. We get $\alpha_1 = 3$ and $\alpha_2 = -1$.

Therefore

$$a_n = 3 * 2^n + (-1)(-1)^n$$

is the solution.

Example: 2

Fibonacci Problem (Follow the white rabbit hoho), find the solution of

$$f_n = f_{n-1} + f_{n-2}, f_0 = 0, f_1 = 1$$

Answer

The characteristic equation is:

$$r^2 - r - 1 = 0$$

Therefore $r_1 = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{5}}{2}$. By the formula, we get

$$f_n = \alpha_1(\frac{1+\sqrt{5}}{2})^n + \alpha_2(\frac{1-\sqrt{5}}{2})^n \tag{5}$$

Substituting $n = 0$ and $n = 1$ in equation 5, we get:

$$f_0 = 0 = \alpha_1 + \alpha_2 \tag{6}$$

$$f_1 = 1 = \alpha_1(\frac{1+\sqrt{5}}{2}) + \alpha_2(\frac{1-\sqrt{5}}{2}) \tag{7}$$

We use equation 6 and equation 7 to get α_1 and α_2. We get $\alpha_1 = \frac{1}{\sqrt{5}}$ and $\alpha_2 = \frac{-1}{\sqrt{5}}$. Therefore the final solution is:

$$f_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$$

Warning! Theorem 1 can be used only when all the roots rs are different, otherwise we cannot find α_i.

Theorem 2

Let c_1 and c_2 be REAL numbers where $c_2 \neq 0$. Suppose $r^2 - c_1 r - c_2 = 0$ has only one root (say, r_0), then

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

has

$$a_n = \alpha_1 r_0^n + \alpha_2 * n * r_0^n \tag{8}$$

as its general solution, where $n = 0, 1, 2, \ldots$ and α_1, α_2 are constants.

We can prove it in the same way as case with distinct roots.

Example: 1

Find the solution of $a_n = 4(a_{n-1} - a_{n-2}), n > 1, a_0 = 0, a_1 = 1$

Answer

The recurrence relation is:

$$a_n = 4a_{n-1} - 4a_{n-2}$$

First, find the characteristic equation, we get:

$$r^2 - 4r + 4 = 0$$

This characteristic equation has two same-value roots, both equal to 2. Therefore, following the formula, we get:

$$a_n = \alpha_1 2^n + \alpha_2 * n * 2^n$$

Substitute $a_0 = 0$ and $a_1 = 1$, we get two equation:

$$0 = \alpha_1 2^0 + \alpha_2 * 0 * 2^0 = \alpha_1$$

and

$$1 = \alpha_1 2^1 + \alpha_2 * 1 * 2^1 = 2\alpha_1 + 2\alpha_2$$

Therefore $\alpha_1 = 0$ and $\alpha_2 = \frac{1}{2}$. This means the solution is:

$$a_n = 2^{-1} * n * 2^n = n * 2^{n-1}$$

Example: 2

Find the solution of $a_n = 3a_{n-1} - 4a_{n-3}, n > 1, a_0 = 0, a_1 = 6, a_2 = 24$

Answer

First, find characteristic equation:

$$0 = r^3 - 3r^2 + 4 = (r-2)(r-2)(r+1)$$

We get $r_1 = 2, r_2 = 2, r_3 = -1$. There are both distinct roots and replicated roots. So we use both formula to get the following general form:

$$a_n = \alpha_1 r_1^n + \alpha_2 * n * r_2^n + \alpha_3 r_3^n \tag{9}$$

Substitute $a_0 = 0$ in equation 9, we get

$$0 = \alpha_1 + \alpha_3 \tag{10}$$

Substitute $a_1 = 6$ in equation 9, we get

$$\begin{aligned} 6 &= \alpha_1 * 2^1 + \alpha_2 * 1 * 2^1 + \alpha_3 * (-1) \\ &= 2\alpha_1 + 2\alpha_2 - \alpha_3 \end{aligned} \tag{11}$$

Substitute $a_2 = 24$ in equation 9, we get

$$\begin{aligned} 24 &= \alpha_1 * 2^2 + \alpha_2 * 2 * 2^2 + \alpha_3 * (-1)^2 \\ &= 4\alpha_1 + 8\alpha_2 + \alpha_3 \end{aligned} \tag{12}$$

From equation 10, 11, and 12, we get the values of alphas.

$$\alpha_1 = 0, \alpha_2 = 3, \alpha_3 = 0$$

Therefore $a_n = 3n2^n, (n \geq 0)$.

$\boxed{\text{Generalization of Theorem 1 and 2}}$

Theorem 3

Let $c_1, c_2, c_3, \ldots, c_k$ be REAL numbers. Suppose characteristic equation $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0$ has k distinct roots (say, $r_1, r_2, r_3, \ldots, r_k$), then

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} \tag{13}$$

has

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_k r_k^n \tag{14}$$

as its general solution, where $n = 0, 1, 2, \ldots$ and $\alpha_1, \ldots, \alpha_k$ are constants.

Example:

Find a general solution of

$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}, a_0 = 2, a_1 = 5, a_2 = 15$

Answer

First, make the characteristic equation and find its roots:

$$\begin{aligned} r^3 - 6r^2 + 11r - 6 &= 0 \\ (r-1)(r-2)(r-3) &= 0 \end{aligned}$$

Roots are $r_1 = 1, r_2 = 2, r_3 = 3$.

Therefore the general solution form, with αs attached, is:

$$a_n = \alpha_1 + \alpha_2 * 2^n + \alpha_3 * 3^n$$

Substitute $a_0 = 2, a_1 = 5, a_2 = 15$, we get 3 equations with 3 unknowns, so all the αs can be found, and the final solution discovered (by you :))

Theorem 4

Let $c_1, c_2, c_3, \ldots, c_k$ be REAL numbers. Suppose characteristic equation $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0$ has t distinct roots (say, $r_1, r_2, r_3, \ldots, r_t$) with multiplicities $m_1, m_2, m_3, \ldots, m_t$ (This implies $m_1 + m_2 + m_3 + \ldots + m_t = k$),

then

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

has

$$\begin{aligned} a_n &= (\alpha_{1,0} + \alpha_{1,1} n + \ldots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ &+ (\alpha_{2,0} + \alpha_{2,1} n + \ldots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ &+ \ldots \\ &+ (\alpha_{t,0} + \alpha_{t,1} n + \ldots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

as its general solution, where $n = 0, 1, 2, \ldots$ and $\alpha_{i,j}$ are constants for $1 \le i \le t$ and $0 \le j \le m_i - 1$.

Solving Non-Homogeneous Recurrence Relation

Example:

$$a_n = 3a_{n-1} + 2n$$

Theorem 5

For recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)$$

The solution is

$$a_n^{(p)} + a_n^{(h)}$$

where $a_n^{(h)}$ is the solution of the Homogeneous part. And $a_n^{(p)}$ can be found from the following pattern:

$f(n)$	$a_n^{(p)}$
c^n	pc^n
$c_t n^t + c_{t-1} n^{t-1} + \ldots + c_1 n + c_0$	$p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0$
$(c_t n^t + c_{t-1} n^{t-1} + \ldots + c_1 n + c_0) c^n$	$(p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0) c^n$
$(c_t n^t + c_{t-1} n^{t-1} + \ldots + c_1 n + c_0) c^n$	$n^m (p_t n^t + p_{t-1} n^{t-1} + \ldots + p_1 n + p_0) c^n$

The last case in the table is for when c is a root of the characteristic equation with multiplicity m.

Example: 1

Find the solution of $a_n = 2a_{n-1} - 6, a_0 = 1$

Answer

First, let us find $a_n^{(p)}$, using the pattern, we get:

$$a_n^{(p)} = p$$

Substitute $a_n^{(p)}$ in the original recurrence relation:

$$p = 2p - 6$$

Therefore, $p = 6 = a_n^{(p)}$.

Ok. We finish with $a_n^{(p)}$. It is time to find $a_n^{(h)}$

The characteristic equation from the original (excluding the non-homogeneous part) is:

$$0 = r - 2r^0 = r - 2$$

Therefore $r = 2$. $a_n^{(h)}$ is $\alpha_1 2^n$.

Substitute $a_0 = 1$ for the whole recurrence relation, we get:

$$1 = a_0^{(p)} + a_0^{(h)} = 6 + \alpha_1 2^0$$

Therefore $\alpha_1 = -5$.

The final solution is $a_n^{(p)} + a_n^{(h)} = 6 + -5 * 2^n$.

Example: 2

Find the solution of $a_n = 2a_{n-1} + (n+1)4^n - 2, a_0 = 5$

Answer

First, let us find $a_n^{(h)}$. Since it is the same as the last example, we get $a_n^{(h)}$ is $\alpha_1 2^n$.

Now we find $a_n^{(p)}$.

By the table, $a_n^{(p)} = (p_0 n + p_1)4^n + p_2$. We substitute this in the original recurrence relation. We get:

$$(p_0 n+p_1)4^n+p_2 = 2(p_0(n-1)+p_1)4^{n-1}+p_2+(n+1)4^n-2$$

We find each of the ps by comparing the coefficients. The above equation can be rearranged as:

$$(0.5p_0)n4^n+(0.5p_1+0.5p_0)4^n+(-p_2) = n4^n+4^n-2$$

We can see that $0.5p_0 = 1$, $0.5p_1 + 0.5p_0 = 1$, and $-p_2 = -2$.

Therefore $p_0 = 2, p_1 = 0, p_2 = 2$. So

$$a_n^{(p)} = (2n)4^n + 2$$

The solution is:

$$a_n = a_n^{(p)} + a_n^{(h)} = 2n4^n + 2 + \alpha_1 2^n$$

We finally find α_1 by substituting the above equation with $a_0 = 5$. We get $\alpha_1 = 3$.

Thus, the final answer will be $a_n = 2n4^n + 2 + 3 * 2^n$.

Example: 3

Find the solution of

$$a_n - 2a_{n-1} = 4 * 2^n, a_0 = 4$$

Answer

First, rearrange, we get $a_n = 2a_{n-1} + 4 * 2^n$. The $a_n^{(h)}$ is the same as before, $\alpha_1 2^n$.

The main thing here is finding $a_n^{(p)}$. From the table:

$$a_n^{(p)} = p_0 * 2^n * n^1$$

The n^1 is multiplied in because 2 is the same as the root value.

Apply this to the original recurrence relation, we get:

$$np_0 * 2^n - 2 * (n-1) * p_0 * 2^{n-1} = 4 * 2^n$$

Solve this, we get $p_0 = 4$.

Now we can find the final solution.

$$a_n = a_n^{(h)} + a_n^{(p)} = \alpha_1 2^n + 4 * 2^n * n$$

Substituting $a_0 = 4$, we get $\alpha_1 = 4$. Therefore the final answer is

$$a_n = 4n * 2^n + 4 * 2^n$$

$\boxed{\text{Finding } a_n^{(p)}}$

Example: 1 Find the $a_n^{(p)}$ of $a_n = 3a_{n-1} + 2n, a_1 = 3$.

Answer

Characteristic equation is $0 = r - 3$, therefore $r = 3$ and $a_n^{(h)} = \alpha * 3^n$.

Now we find $a_n^{(p)}$ from $2n$.

$$a_n^{(p)} = p_0 n + p_1$$

Substitute $a_n^{(p)}$ in the original recurrence relation. We can find p_0 and p_1 by comparing coefficients:

$$\begin{aligned} p_0 n + p_1 &= 3(p_0(n-1) + p_1) + 2n \\ p_0 n + p_1 &= 3p_0 n - 3p_0 + 3p_1 + 2n \\ 0 &= 2p_0 n - 3p_0 + 2p_1 + 2n \\ 0 &= n(2p_0 + 2) + (2p_1 - 3p_0) \end{aligned}$$

$2p_0 + 2$ and $2p_1 - 3p_0$ must be 0. Therefore we get $p_0 = -1$ and $p_1 = -1.5$.

Thus $a_n^{(p)} = -n - 1.5$.

Example: 2

Write the $a_n^{(p)}$ of a recurrence relation whose roots of characteristic equation are $r_1 = r_2 = 3$, where:

- $f(n) = 3^n$

  Answer

  $$a_n^{(p)} = p * 3^n * n^2$$

  because 3 is the same value as the 2 roots.

- $f(n) = n * 3^n$

  Answer

  $$a_n^{(p)} = (p_1 n + p_0) * 3^n * n^2$$

- $f(n) = n^2 * 2^n$

  Answer

  $$a_n^{(p)} = (p_0 n^2 + p_1 n + p_2) * 2^n$$

- $f(n) = (n^2 + 1) * 3^n$

  Answer

  $$a_n^{(p)} = (p_0 n^2 + p_1 n + p_2) * 3^n * n^2$$

Example: 3

Find $a_n^{(p)}$ of $a_n = a_{n-1} + n$.

Answer

First, the characteristic equation

$$0 = r - 1, r = 1$$

Therefore $a_n^{(h)}$ is $\alpha * 1^n = \alpha$.

Notice that $f(n) = n$ is actually $f(n) = n * 1^n$. i.e. it has the root.

Therefore

$$\begin{aligned} a_n^{(p)} &= (p_0 n + p_1) * 1^n * n^1 \\ &= n(p_0 n + p_1) \end{aligned}$$

Substitute it in the original recurrence relation, we get:

$$\begin{aligned} n(p_0 n + p_1) &= (n-1)(p_0(n-1) + p_1) + n \\ p_0 n^2 + p_1 n &= p_0 n^2 - p_0 n + p_1 n - p_0 n + p_0 - p_1 + n \\ 0 &= n(1 - 2p_0) + (p_0 - p_1) \end{aligned}$$

Therefore $p_0 = p_1 = 0.5$. $a_n^{(p)} = n(0.5n + 0.5)$.